Numerical integration of partial differential equations

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Outline

I.- Diffusion Models

1. Diffusion models

II.- Age-Structured Population Models

2. Bounded Mortality — Infinite age

III.- Size-Structured Population Models

3. Non Hierarchical models

Part I

Diffusion Models

Outline

1. Diffusion models
   - Models and Numerical analysis

General Model

$$\frac{\partial u}{\partial t} + \mathbf{v}(x,t)\,\nabla\, u = A\,\Delta\, u + f(x,t), \quad (x,t) \in D \times (0,T]$$

- Spatial domain: $D \subset \mathbb{R}^d$, d=1,2,3.
- Diffusion term (Laplace operator): $\Delta\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.
- Velocity field: $\mathbf{v}(x,t)$.
- Reaction term: $f(x,t)$.

Numerical methods for linear parabolic problems

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= A\,\Delta\, u + f(x,t), \quad (x,t) \in D \times (0,T] \\
u(\mathbf{x},0) &= u_0(\mathbf{x}), \quad \mathbf{x} \in D, \\
u|_{\partial D} &= g(\mathbf{x},t), t > 0.
\end{aligned}
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- Diffusion term (Laplace operator): $\Delta\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.
- Forcing function: $f(\mathbf{x},t)$.
- Value of u on the boundary ∂D: $g(\mathbf{x},t)$.

Hypothesis

- $f(\mathbf{x}, t)$, $g(\mathbf{x}, t)$ are sufficiently smooth.
- $\bar{D} = D \cup \partial D \equiv [a, b] \times [c, d]$

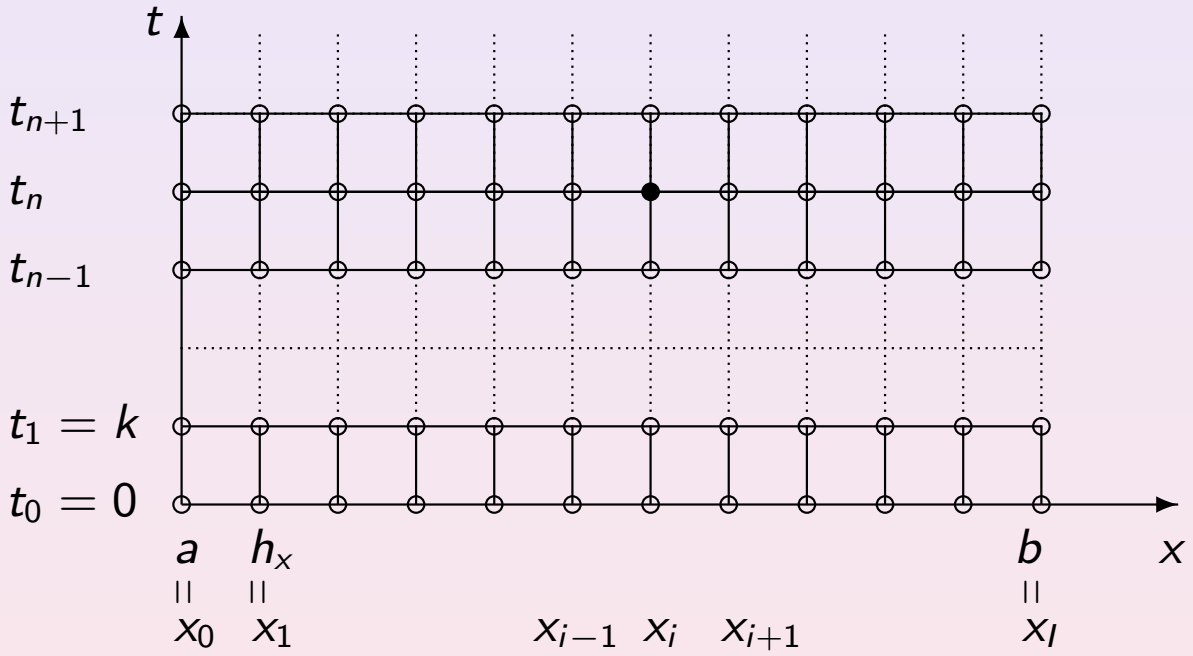
Numerical Parameters

- h_x,h_y, $I = \dfrac{b-a}{h_x}$, $J = \dfrac{d-c}{h_y}$.
  $x_i = a + h_x\, i$, $y_j = c + h_y\, j$.
  $\bar{D}_h = \{(x_i, y_j) \in \bar{D} : 0 \leq i \leq I, 0 \leq j \leq J\}$
- k, constant time step, $\quad N = Tk$.

Notation

- $u_{ij}^n = u(x_i, y_j, t_n)$

Finite difference operators



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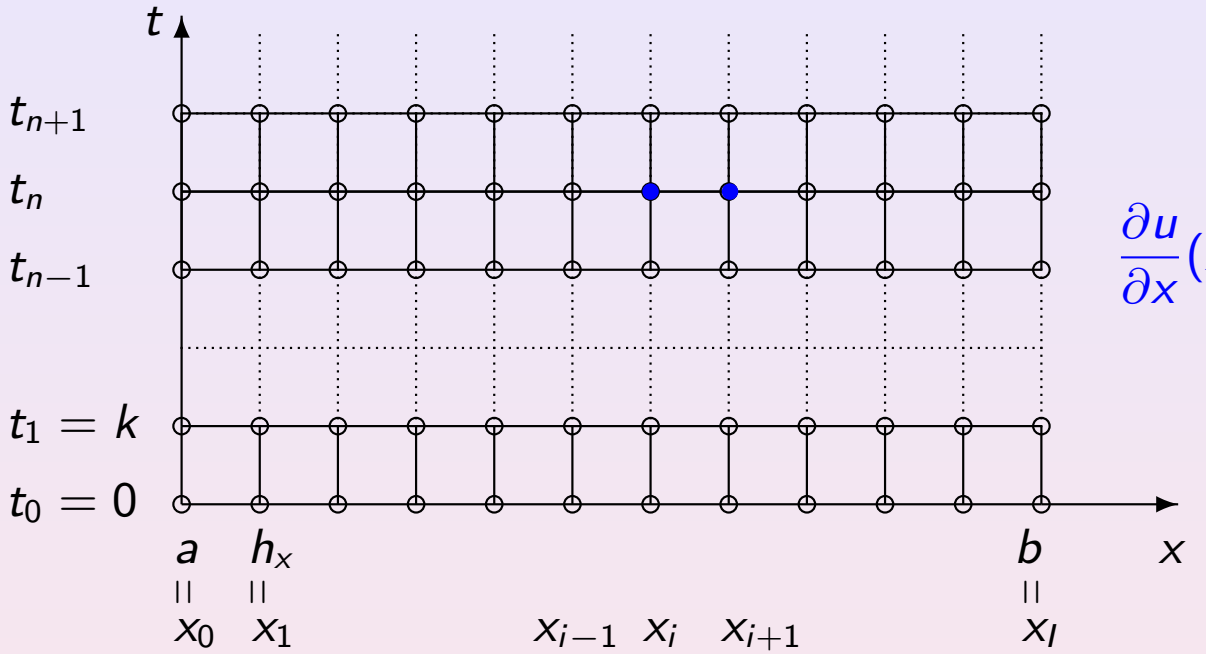

$$\frac{\partial u}{\partial x}(x_i, y_j, t_n) \quad \frac{\partial u}{\partial y}(x_i, y_j, t_n)$$

Forward difference operators

- $\Delta_x^+ u_{ij}^n = u_{i+1,j}^n - u_{ij}^n,$
- $\Delta_y^+ u_{ij}^n = u_{i,j+1}^n - u_{ij}^n,$
- $\Delta_t^+ u_{ij}^n = u_{ij}^{n+1} - u_{ij}^n,$

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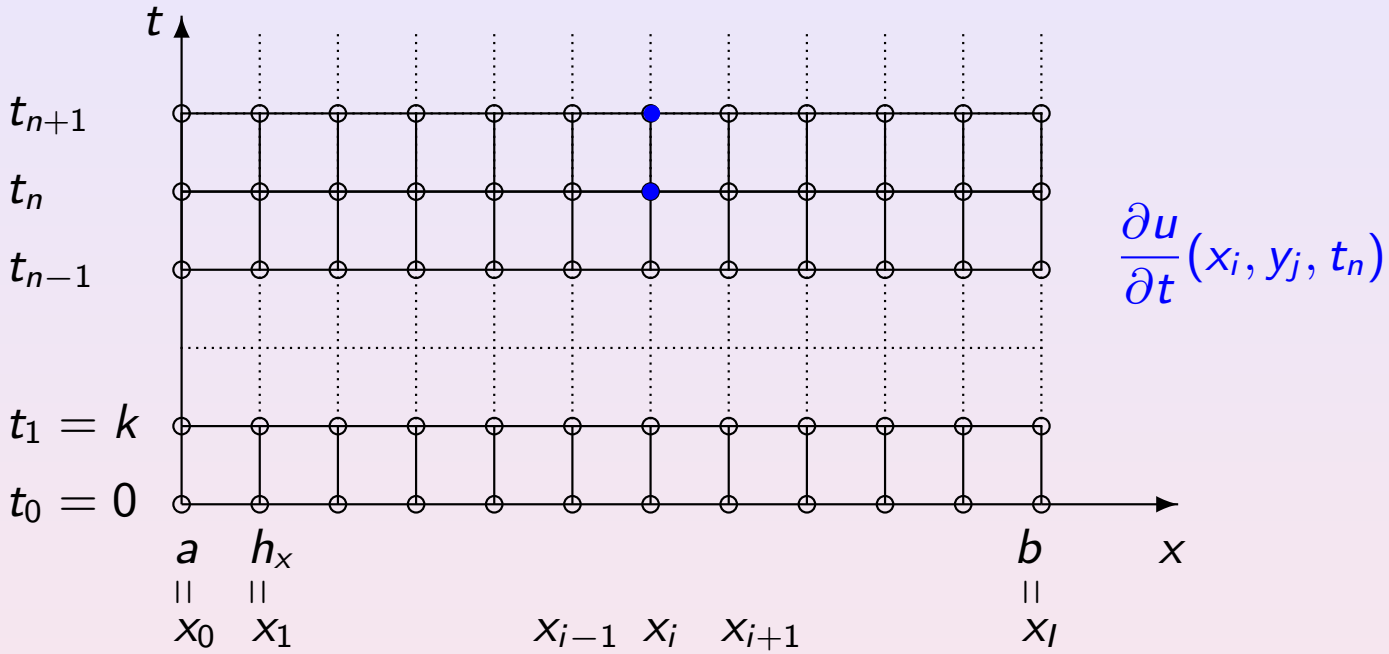

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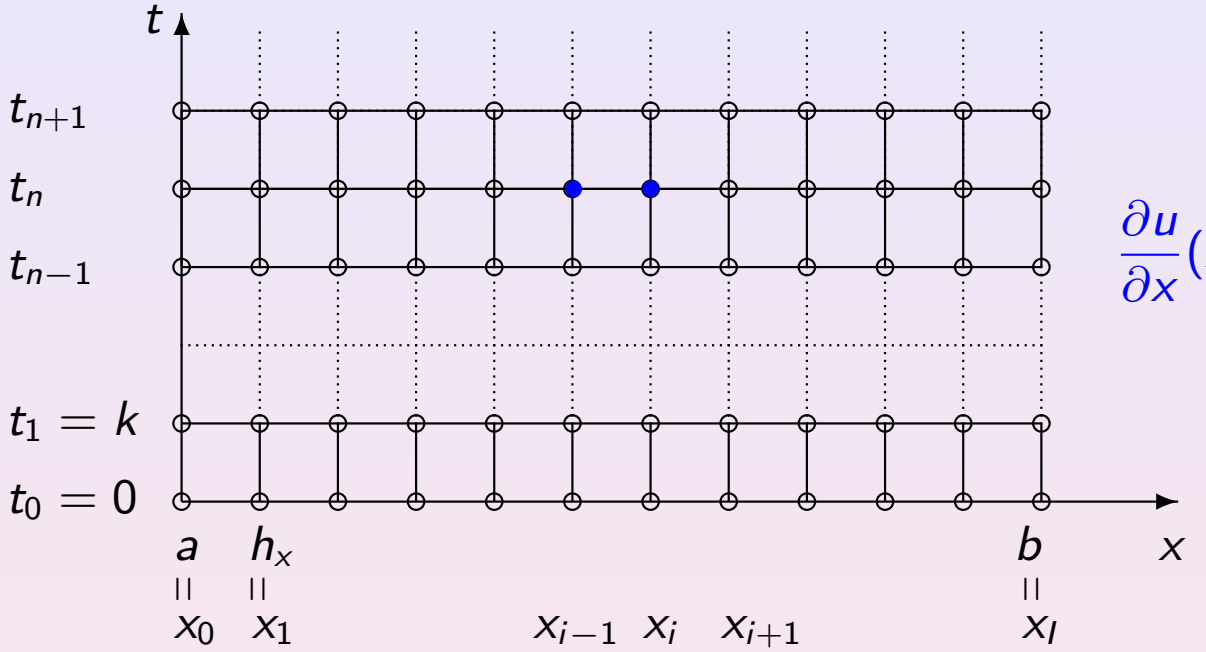

$$\frac{\partial u}{\partial x}(x_i, y_j, t_n) \quad \frac{\partial u}{\partial y}(x_i, y_j, t_n)$$

Backward difference operators

- $\Delta_x^- \, u_{ij}^n = u_{ij}^n - u_{i-1,j}^n$,
- $\Delta_y^- \, u_{ij}^n = u_{ij}^n - u_{i,j-1}^n$,
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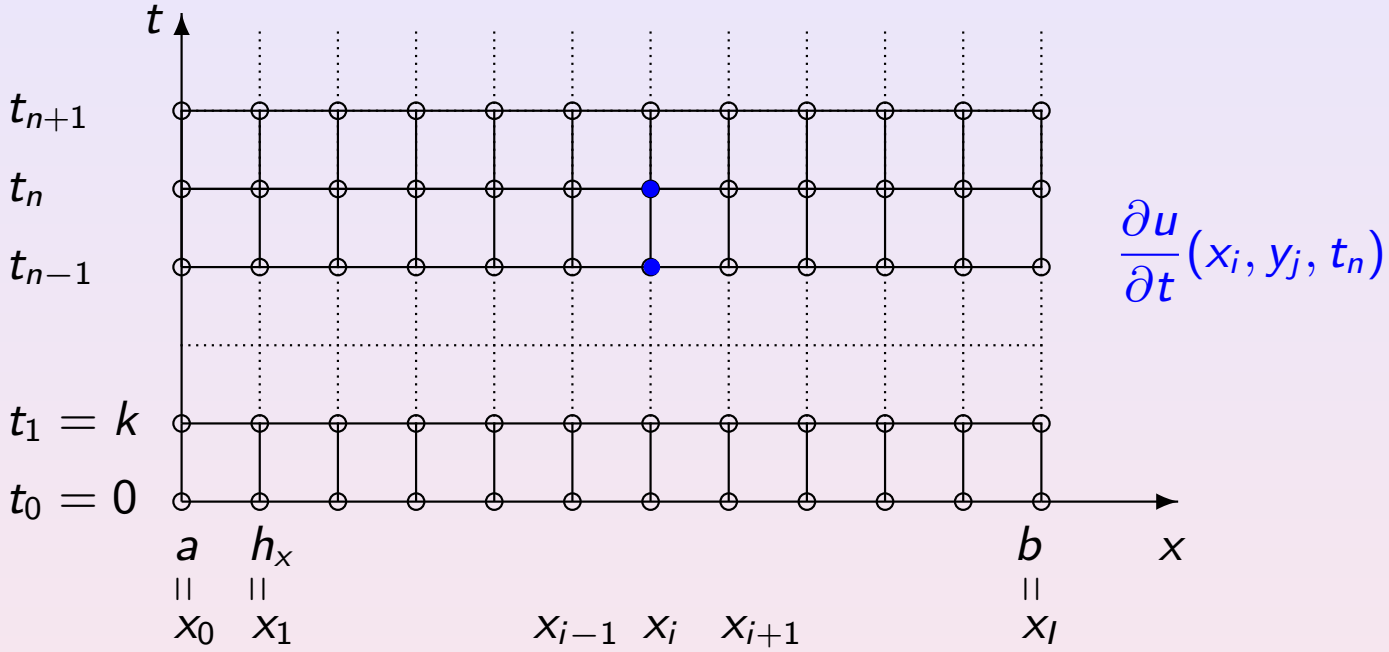

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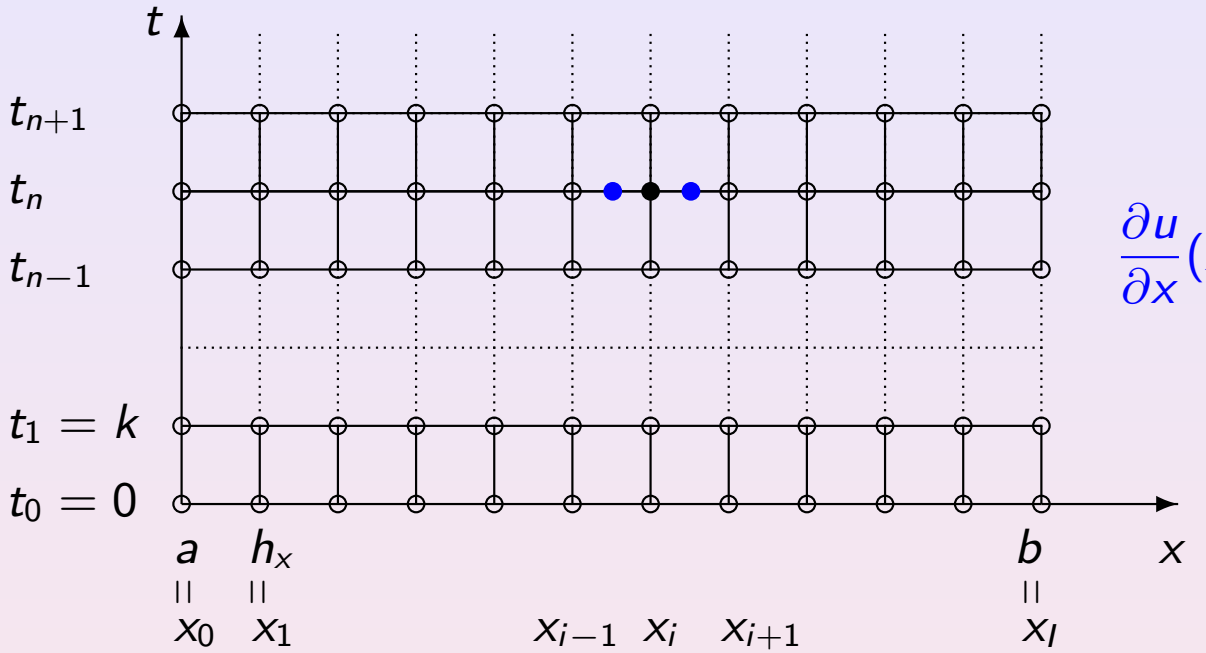
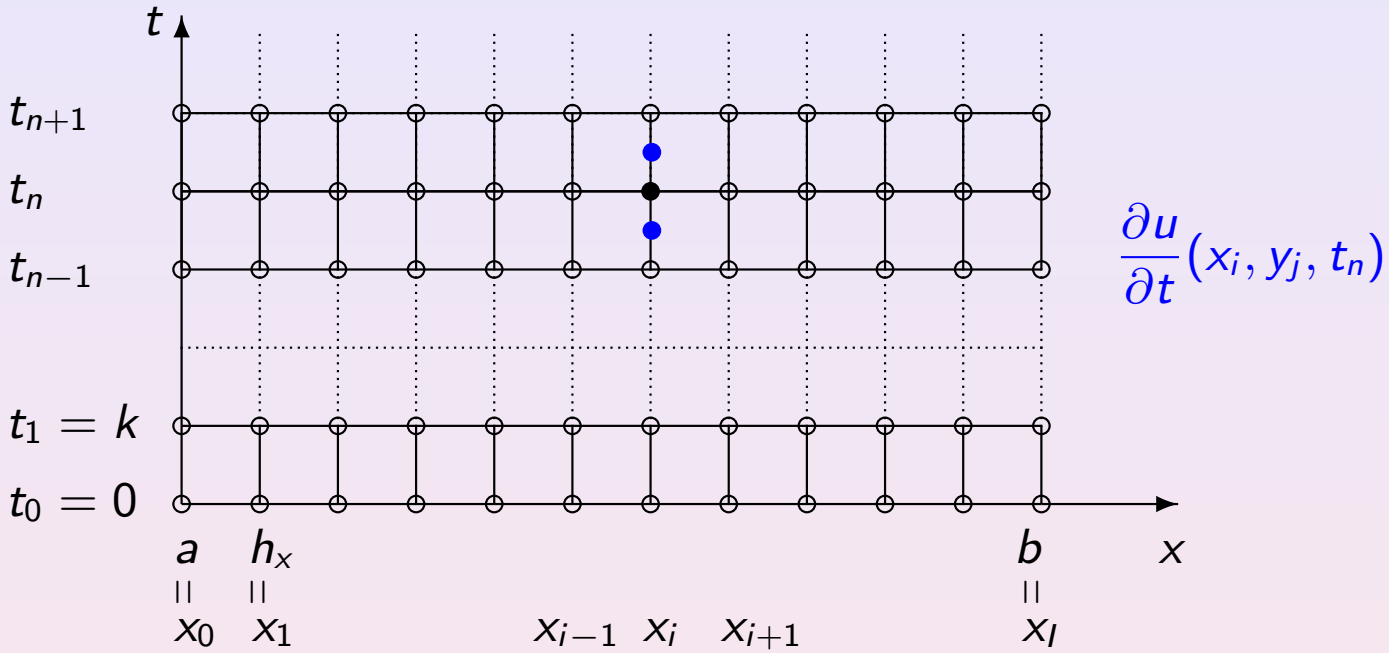


$$\frac{\partial u}{\partial x}(x_i, y_j, t_n) \quad \frac{\partial u}{\partial y}(x_i, y_j, t_n)$$

Central difference operators

- $\delta_x \, u_{ij}^n = u_{i+\frac{1}{2},j}^n - u_{i-\frac{1}{2},j}^n$
- $\delta_y \, u_{ij}^n = u_{i,j+\frac{1}{2}}^n - u_{i,j-\frac{1}{2}}^n$
- $\delta_t \, u_{ij}^n = u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n-\frac{1}{2}}$

Finite difference operators



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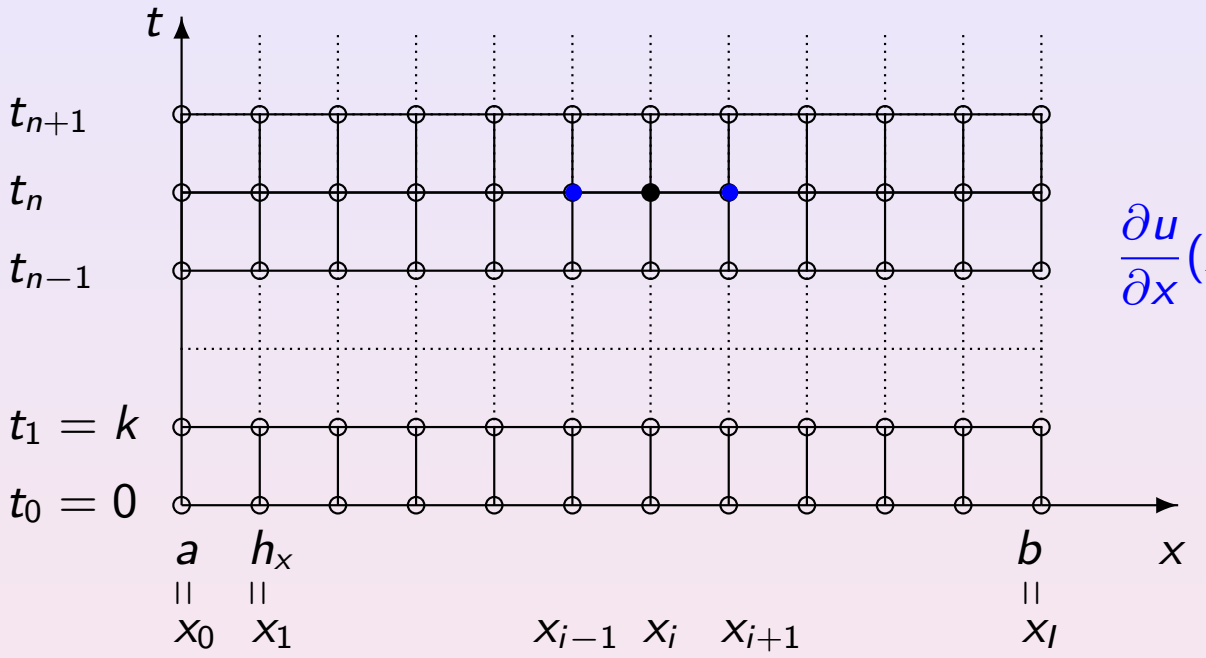

$$\frac{\partial u}{\partial x}(x_i, y_j, t_n) \quad \frac{\partial u}{\partial y}(x_i, y_j, t_n)$$

Doble central difference operators

- $\Delta^+_{0x}\, u^n_{ij} = u^n_{i+1,j} - u^n_{i-1,j}$
- $\Delta^+_{0y}\, u^n_{ij} = u^n_{i,j+1} - u^n_{i,j-1}$
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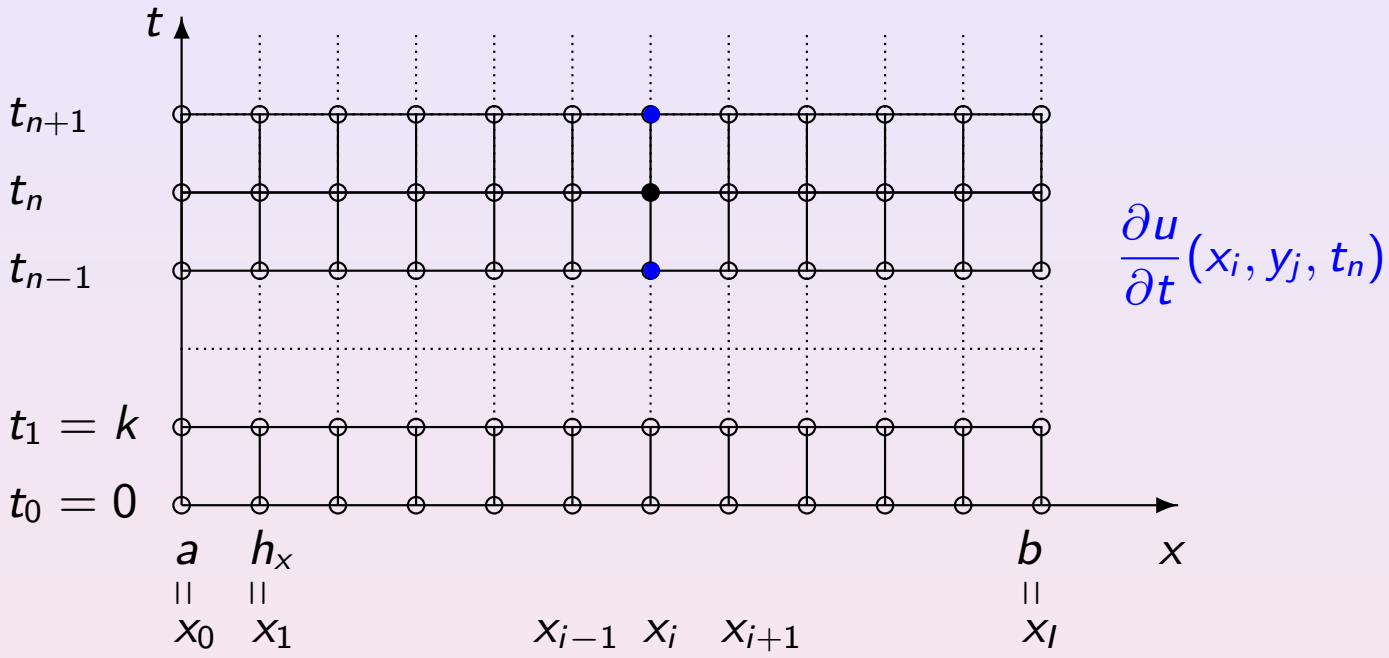

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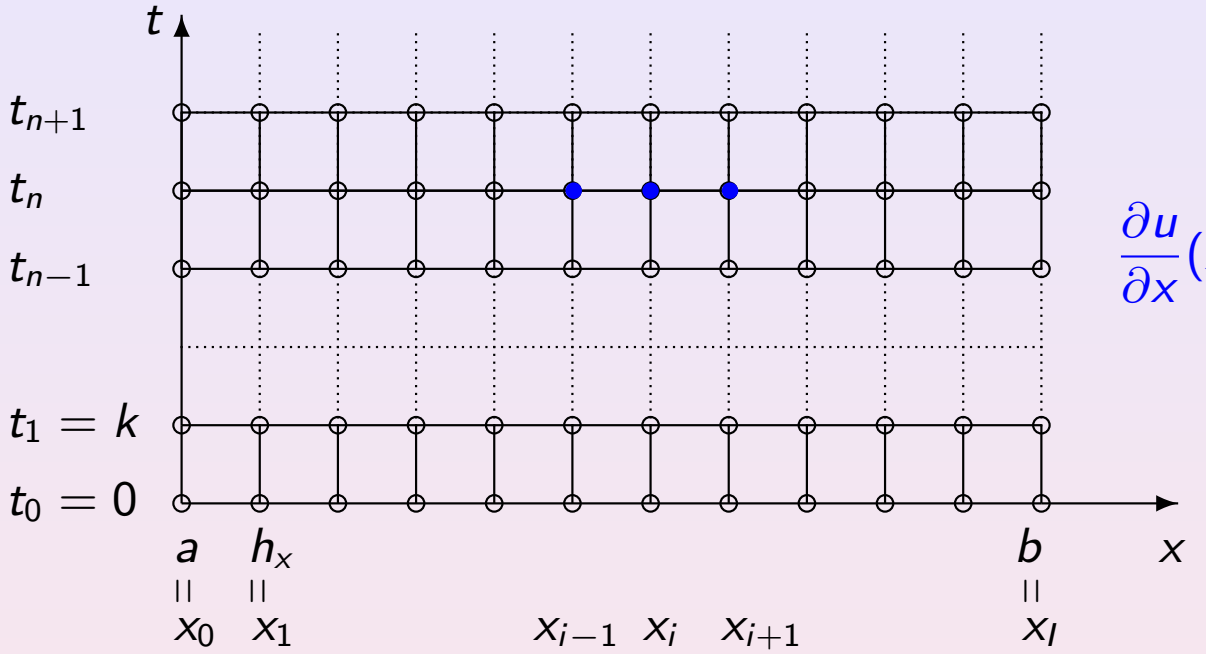
Finite difference operators



$$\frac{\partial u}{\partial x}(x_i, y_j, t_n) \quad \frac{\partial u}{\partial y}(x_i, y_j, t_n)$$

Second order central difference operators

- $\delta_x^2 \, u_{ij}^n = u_{i+1,j}^n - 2\, u_{ij}^n + u_{i-1,j}^n$
- $\delta_y^2 \, u_{ij}^n = u_{i,j+1}^n - 2\, u_{ij}^n + u_{i,j-1}^n$
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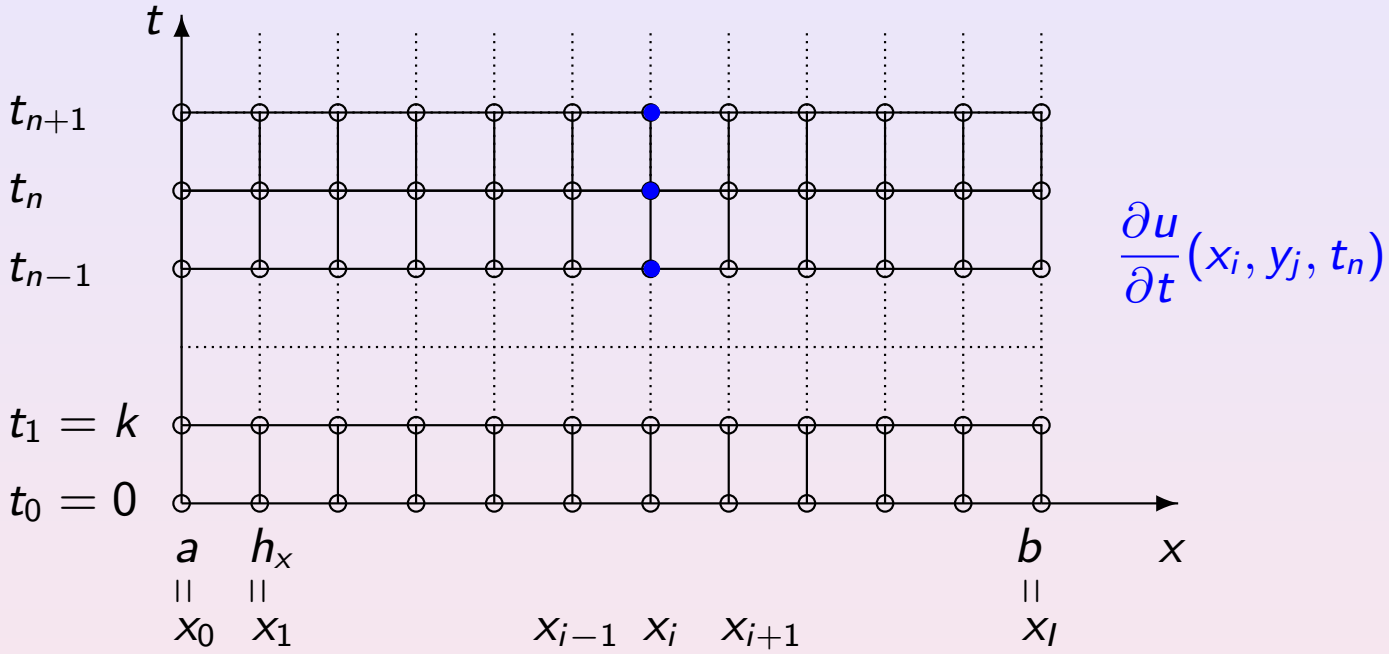
Finite difference operators



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Taylor series expansions

- $\Delta_x^+ \, u_{ij}^n = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_i, y=y_j, t=n\,k} + O(h)$
- $\Delta_x^- \, u_{ij}^n = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_i, y=y_j, t=n\,k} + O(h)$
- $\delta_x \, u_{ij}^n = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_i, y=y_j, t=n\,k} + O(h^2)$
- $\Delta_{0x} \, u_{ij}^n = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_i, y=y_j, t=n\,k} + O(h^2)$
- $\delta_x^2 \, u_{ij}^n = \left.\dfrac{\partial^2 u}{\partial x^2}\right|_{x=x_i, y=y_j, t=n\,k} + O(h^2)$

Euler explicit scheme (EFTCS)

Equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= A\,\Delta\, u + f(x,t), \quad (x,t) \in D \times (0,T] \\
u(\mathbf{x},0) &= u_0(\mathbf{x}), \quad \mathbf{x} \in D, \\
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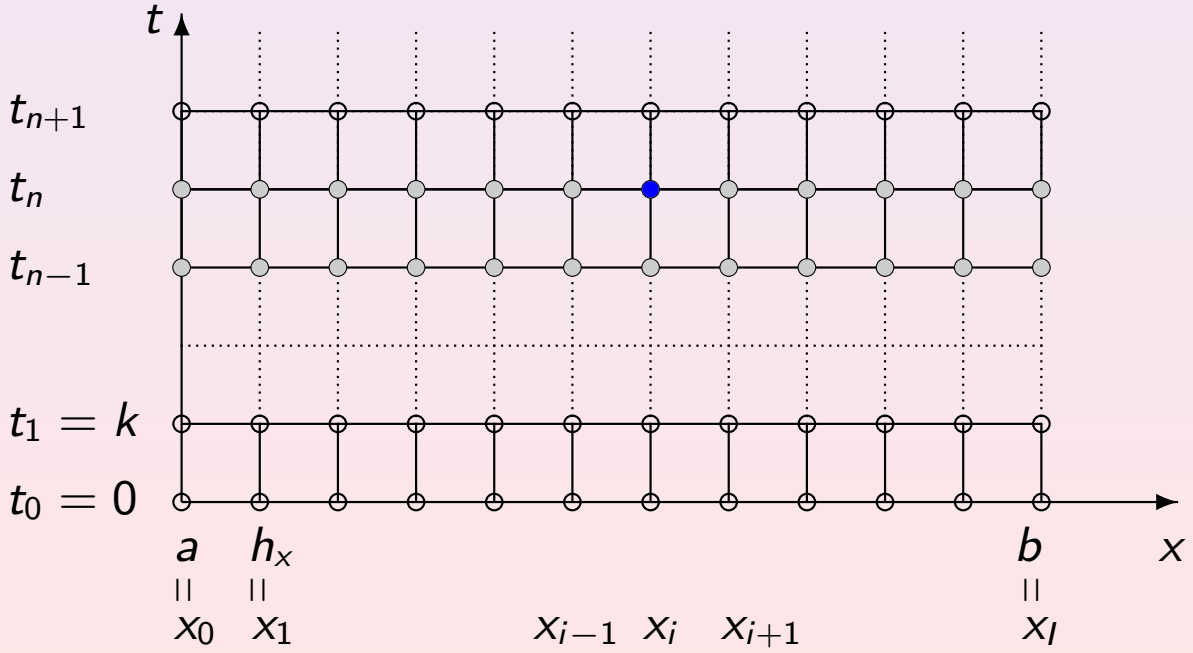

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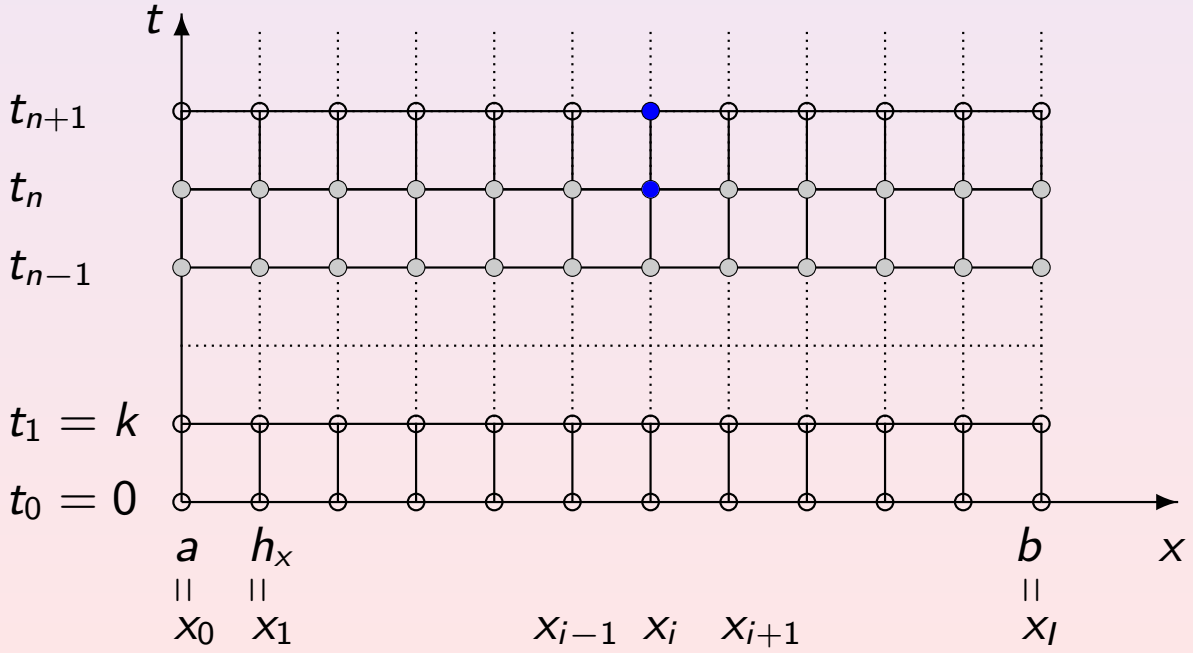

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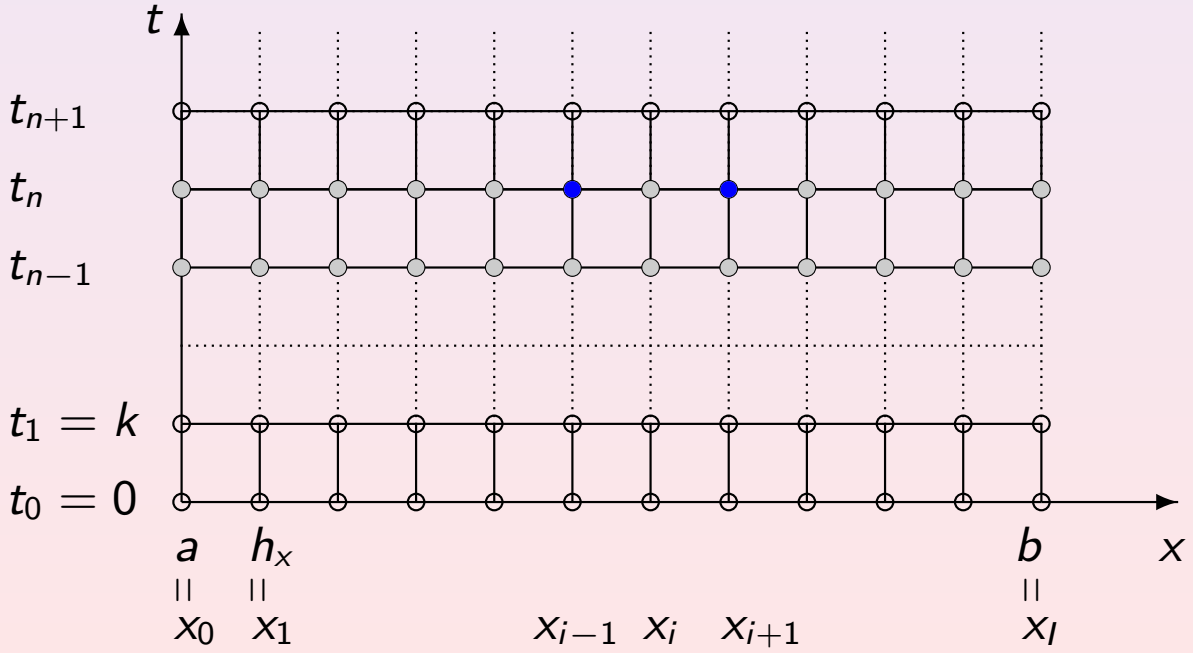

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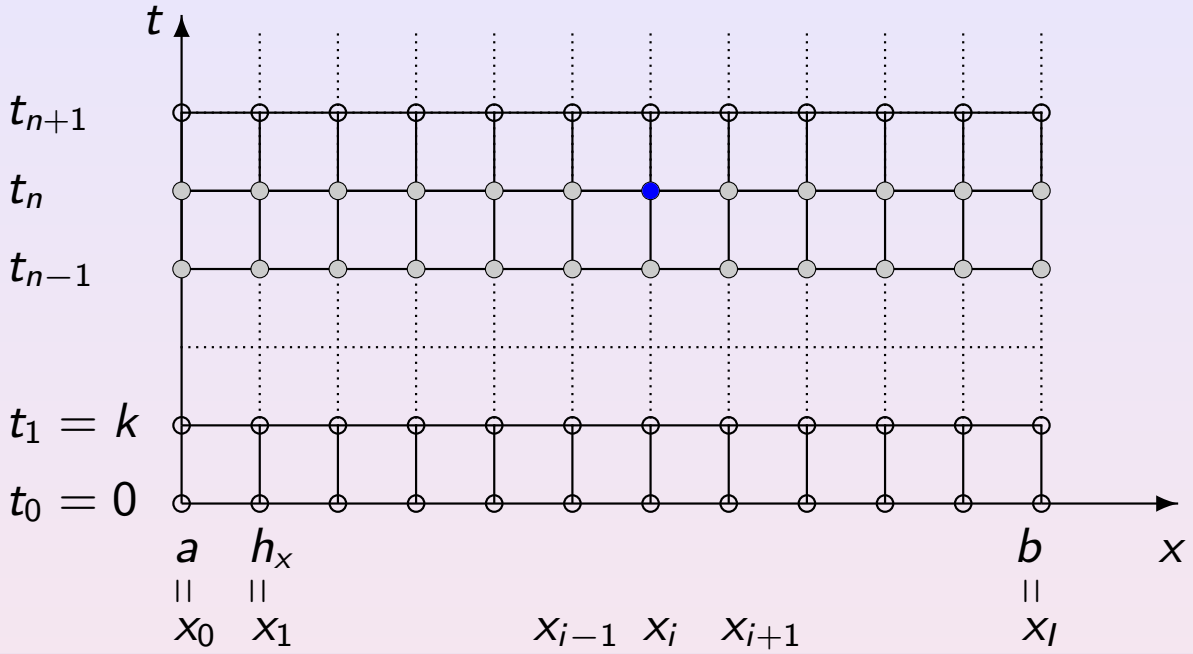

$$
\begin{aligned}
U_{ij}^{n+1} &= U_{ij}^n + A\left(r_x\,\delta_x^2 + r_y\,\delta_y^2\right) U_{ij}^n + k\,f_{ij}^n \\
&\qquad 0 \le i \le I, \quad 0 \le j \le J, \quad 0 \le n \le N. \\
U_{0j}^n &= g(a, y_j, t_n), \quad U_{Ij}^n = g(b, y_j, t_n), \qquad 0 \le i \le I \\
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$$r_x = A\frac{k}{h_x^2},\ r_y = A\frac{k}{h_y^2}$$

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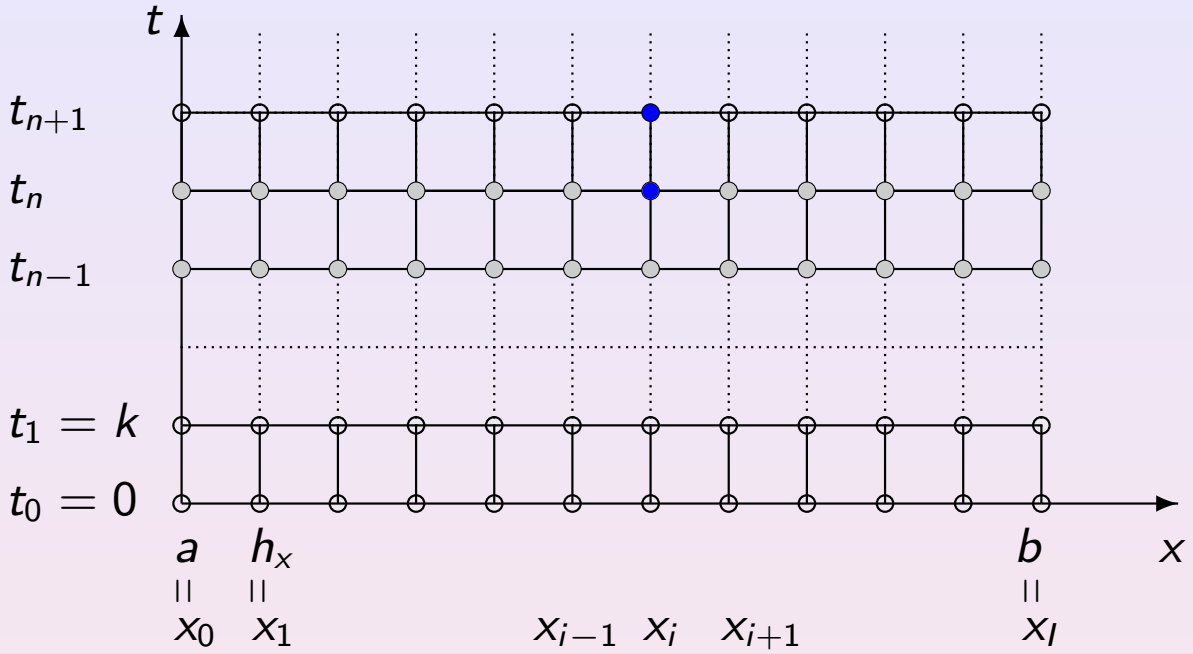

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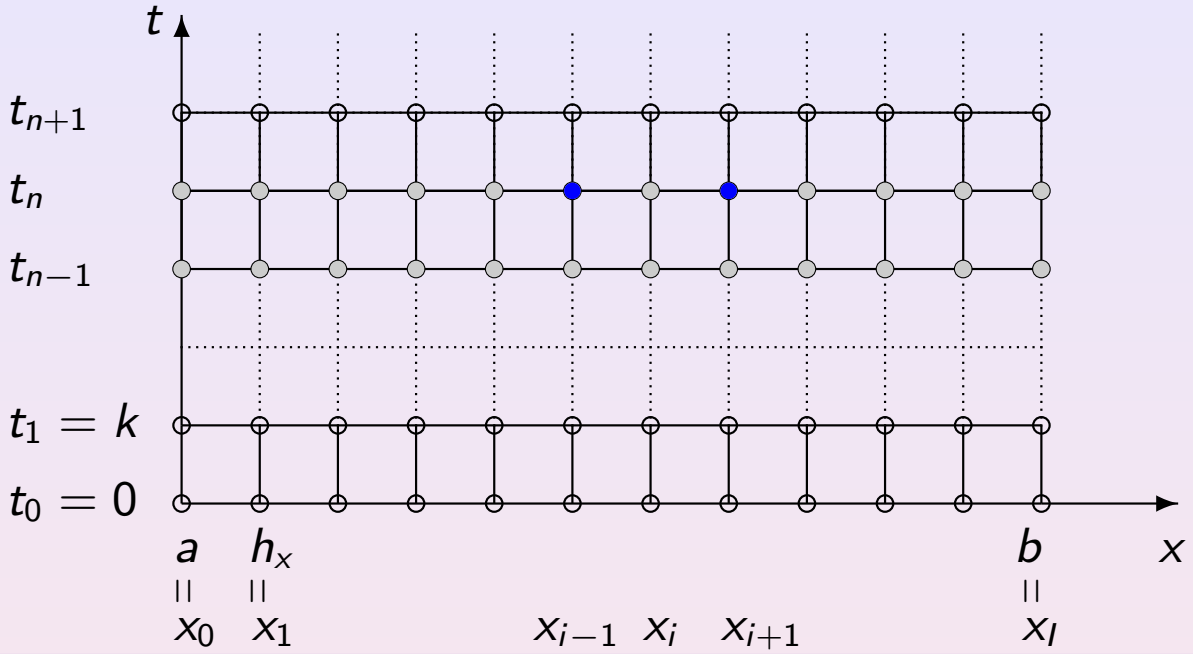

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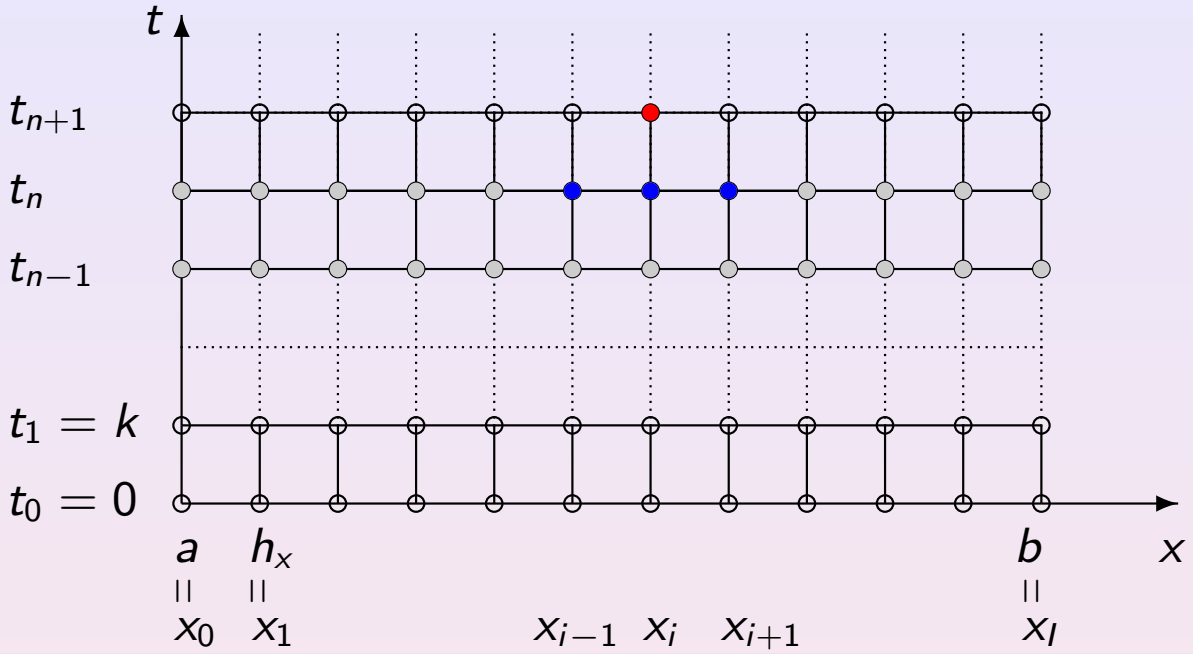

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Convergence analysis.

Truncation error. Taylor expansion. Consistency

$$\tau_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{k} - A\left(\frac{\delta_x^2}{h_x^2} + \frac{\delta_y^2}{h_y^2}\right) u_{ij}^n - f_{ij}^n$$

$$\|\tau_{ij}^n\|_\infty \leq C\,(k + h_x^2 + h_y^2)$$

Consistent of order 1 in time 2 in space.

Stability ($\lambda(\mathbf{w}) \leq 1$). Von neumann analysis

We set $U_{pq}^n \equiv \lambda^n\, e^{i\,(w_x\, h_x + w_y\, h_y)}$. The amplification factor is

$$\lambda(\mathbf{w}) = 1 - 4\left(r_x\, \sin^2\left(\frac{w_x}{2}\right) h_x + r_y\, \sin^2\left(\frac{w_y}{2}\right) h_y\right)$$

$$r_x + r_y \leq \frac{1}{2}$$

Convergence analysis.

Convergence

$$E_{ij}^n = u_{ij}^n - U_{ij}^n$$

$$E_{ij}^{n+1} = (1 - 2\,r_x - 2\,r_y)\,E_{ij}^n + r_x\,(E_{i+1,j}^n + E_{i-1,j}^n) + r_y\,(E_{i,j+1}^n + E_{i,j-1}^n) + k\,\tau_{ij}^n$$

Stability and Consistency properties give us

$$\|E_{ij}^{n+1}\|_\infty \leq \|E_{ij}^0\|_\infty + C\,T\,(h_x^2 + h_y^2 + k).$$

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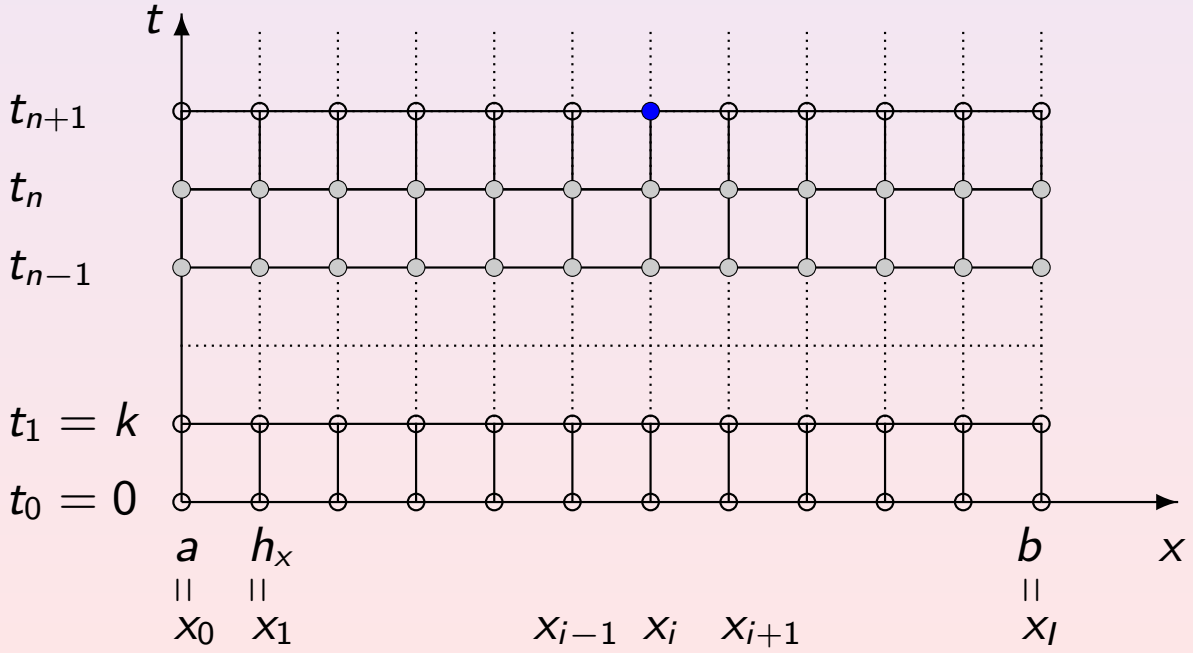

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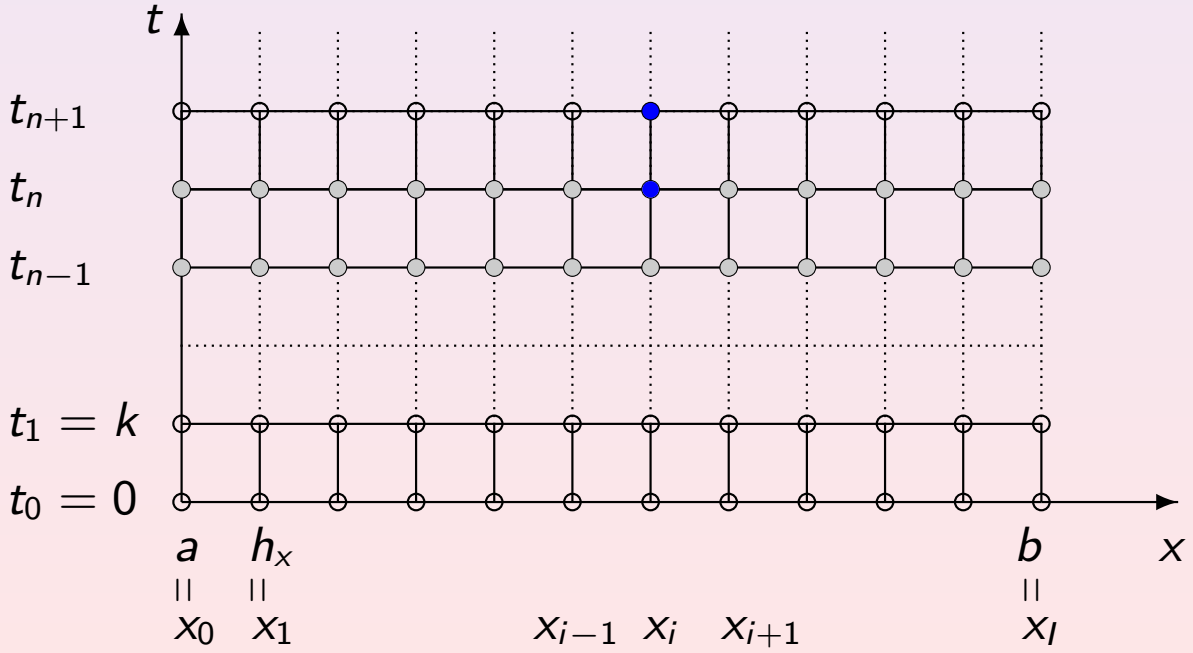

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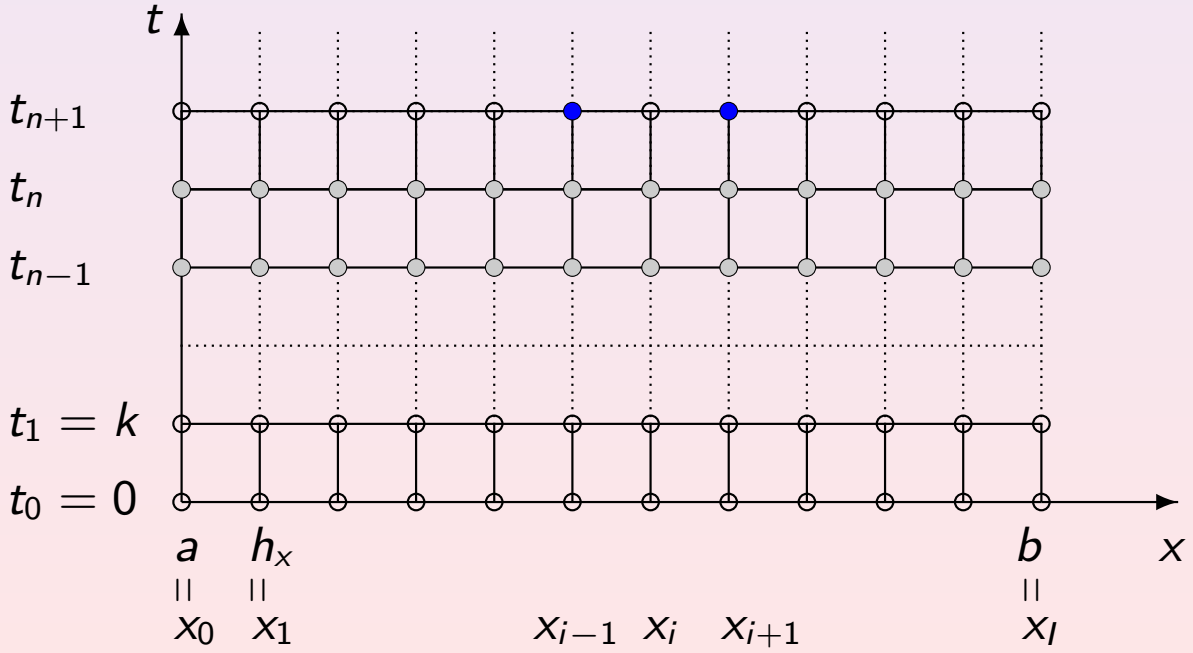

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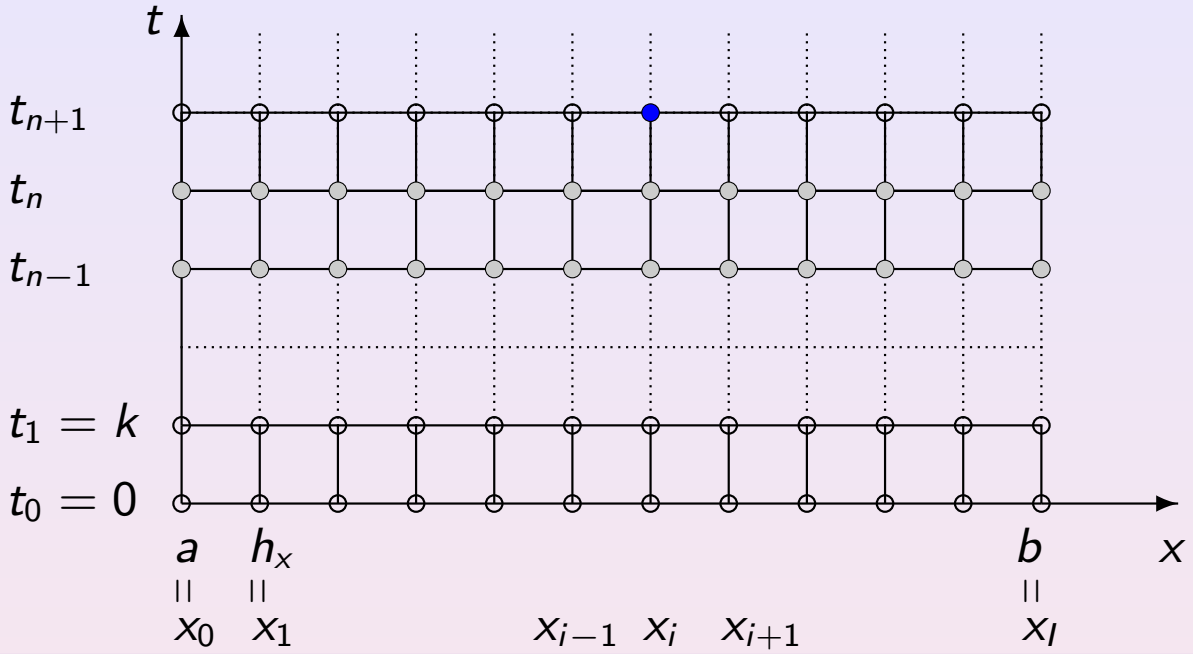

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Unconditionally stable and convergent.

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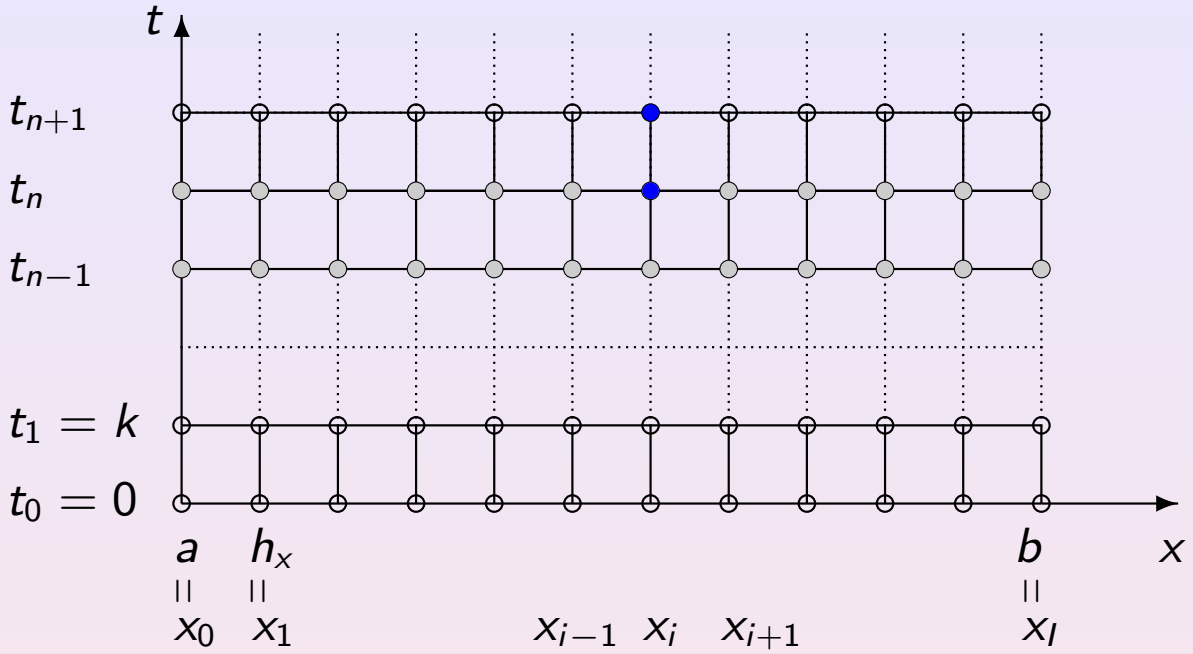

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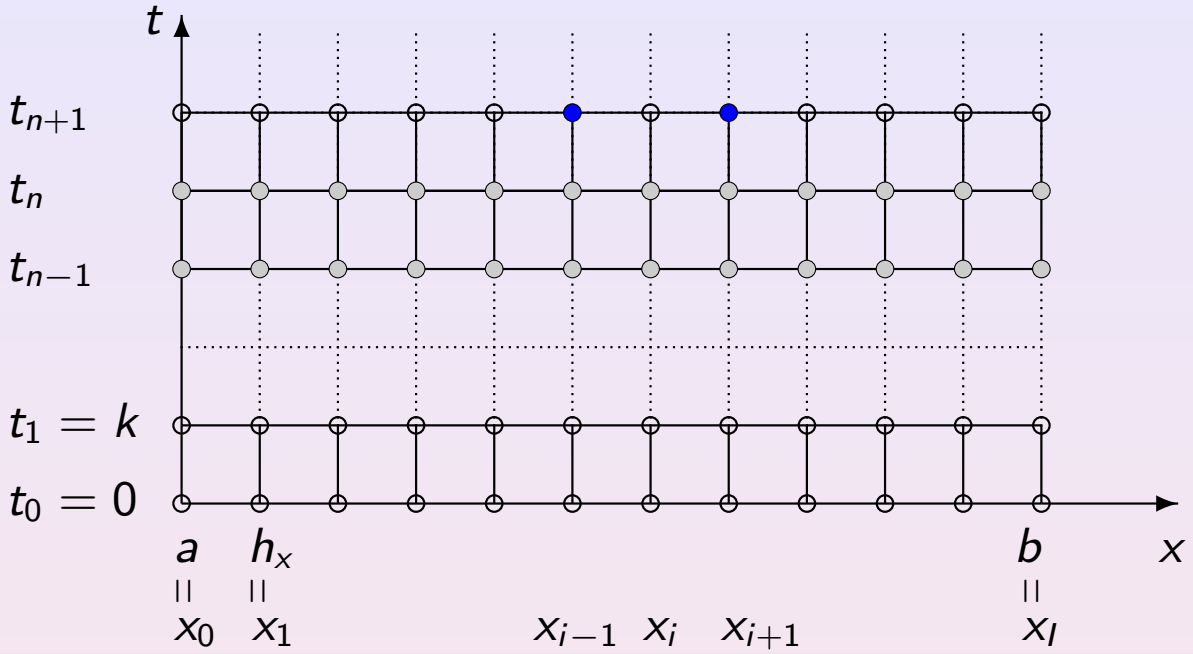
$$0 \le i \le I, \quad 0 \le j \le J, \quad 0 \le n \le N.$$

$$U_{0j}^{n} = g(a, y_j, t_n), \quad U_{Ij}^{n} = g(b, y_j, t_n), \qquad 0 \le i \le I$$

$$U_{i0}^{n} = g(x_i, c, t_n), \quad U_{iJ}^{n} = g(x_i, d, t_n), \qquad 0 \le j \le J$$

Unconditionally stable and convergent.

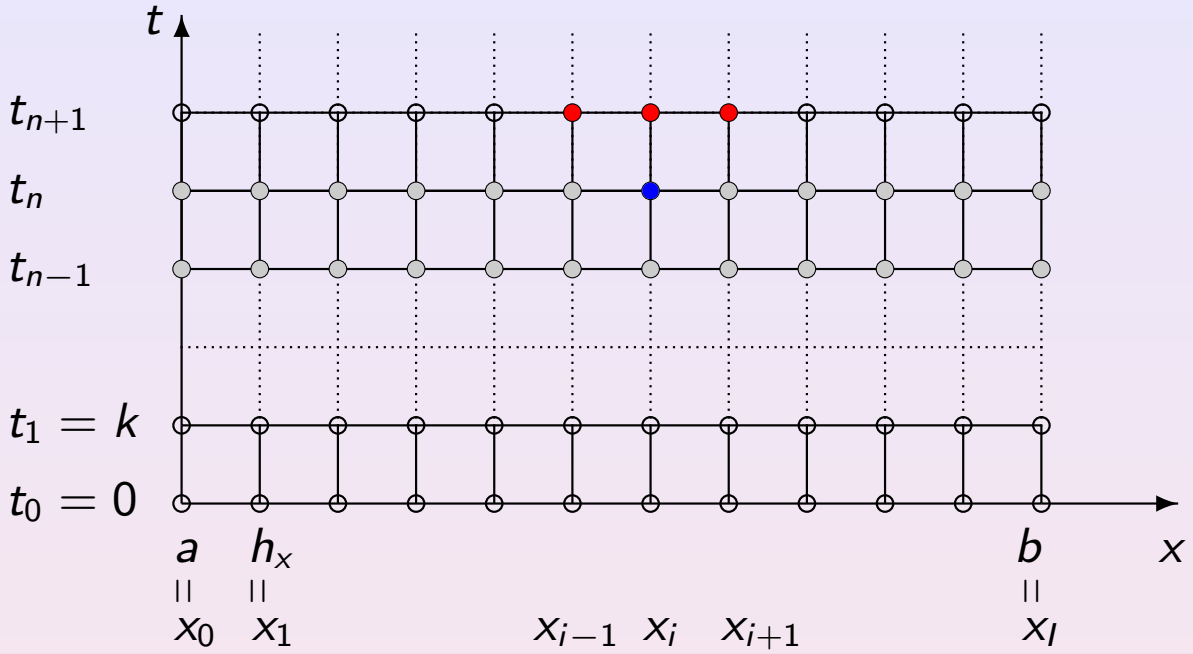
Euler implicit scheme (IBTCS)



$$
\begin{aligned}
U_{ij}^{n+1} \quad - \quad & A\left(r_x\,\delta_x^2 + r_y\,\delta_y^2\right) U_{ij}^{n+1} = U_{ij}^n + k\,f_{ij}^n,\ r_x = A\frac{k}{h_x^2}, r_y = A\frac{k}{h_y^2} \\
& 0 \le i \le I, \quad 0 \le j \le J, \quad 0 \le n \le N. \\
U_{0j}^n \quad = \quad & g(a, y_j, t_n), \quad U_{Ij}^n = g(b, y_j, t_n), \qquad 0 \le i \le I \\
U_{i0}^n \quad = \quad & g(x_i, c, t_n), \quad U_{iJ}^n = g(x_i, d, t_n), \qquad 0 \le j \le J
\end{aligned}
$$

Unconditionally stable and convergent.

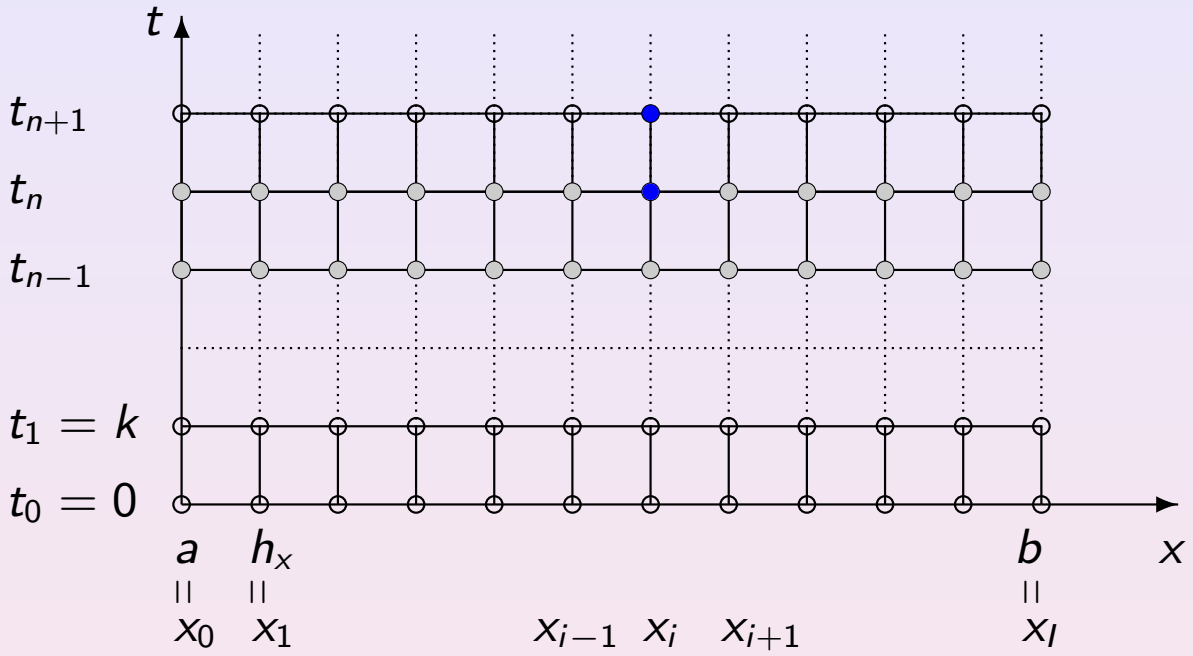
Euler implicit scheme (IBTCS)



$$
\begin{aligned}
U_{ij}^{n+1} \quad - \quad & A\left(r_x\,\delta_x^2 + r_y\,\delta_y^2\right) U_{ij}^{n+1} = U_{ij}^n + k\,f_{ij}^n,\ r_x = A\frac{k}{h_x^2}, r_y = A\frac{k}{h_y^2} \\
& 0 \le i \le I, \quad 0 \le j \le J, \quad 0 \le n \le N. \\
U_{0j}^n \quad = \quad & g(a, y_j, t_n), \quad U_{Ij}^n = g(b, y_j, t_n), \qquad 0 \le i \le I \\
U_{i0}^n \quad = \quad & g(x_i, c, t_n), \quad U_{iJ}^n = g(x_i, d, t_n), \qquad 0 \le j \le J
\end{aligned}
$$

Unconditionally stable and convergent.

Crank-Nicolson Scheme (CN)

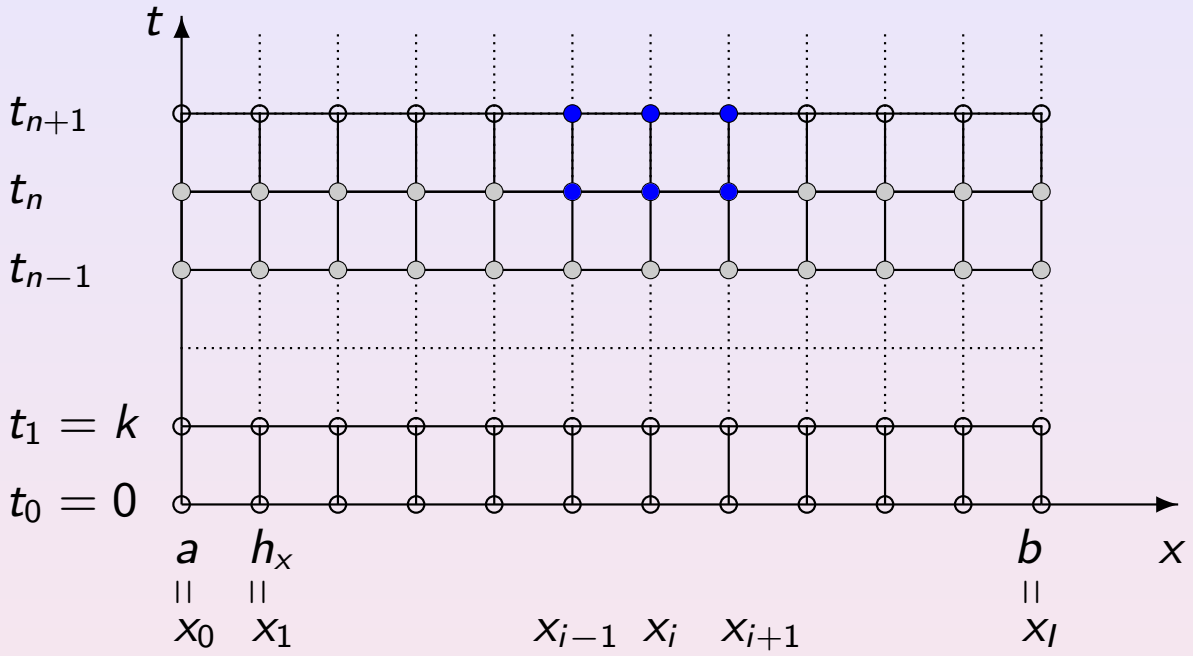


$$U_{ij}^{n+1} - \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n+1} = U_{ij}^{n} + \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n} + \frac{k}{2} \left(f_{ij}^{n} + f_{ij}^{n+1} \right)$$

$$0 \leq i \leq I, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N.$$

$$r_x = A \frac{k}{h_x^2}, r_y = A \frac{k}{h_y^2}$$

Crank-Nicolson Scheme (CN)



$$U_{ij}^{n+1} - \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n+1} = U_{ij}^{n} + \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n} + \frac{k}{2} \left(f_{ij}^{n} + f_{ij}^{n+1} \right)$$

$$0 \leq i \leq I, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N.$$

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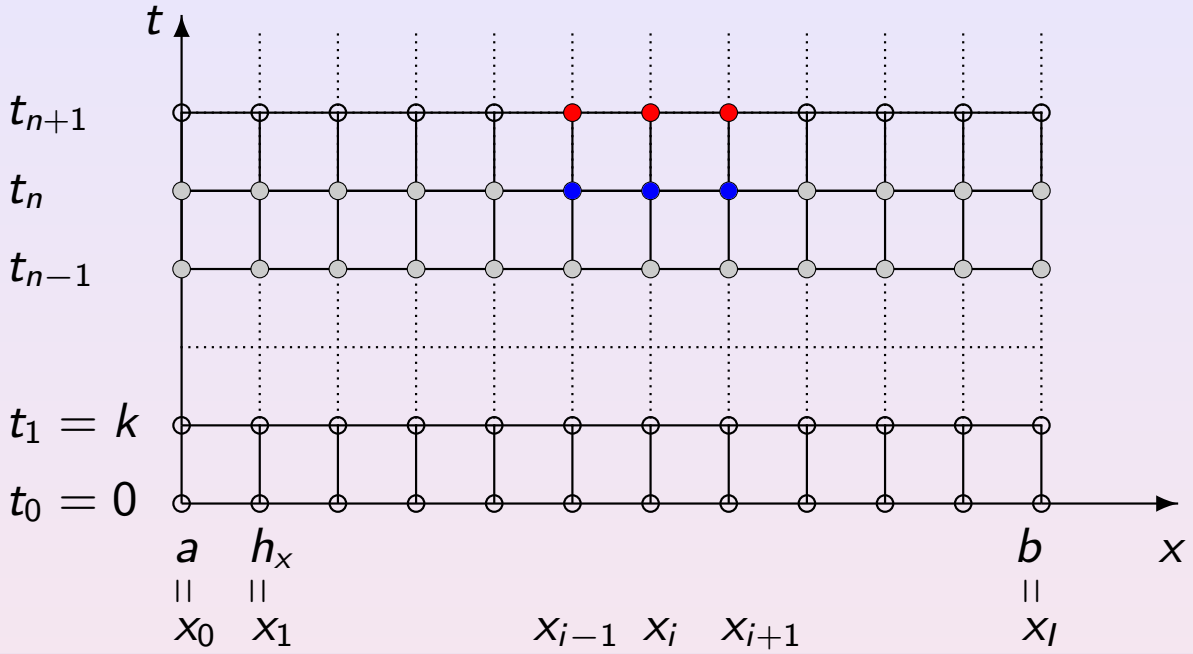
Crank-Nicolson Scheme (CN)



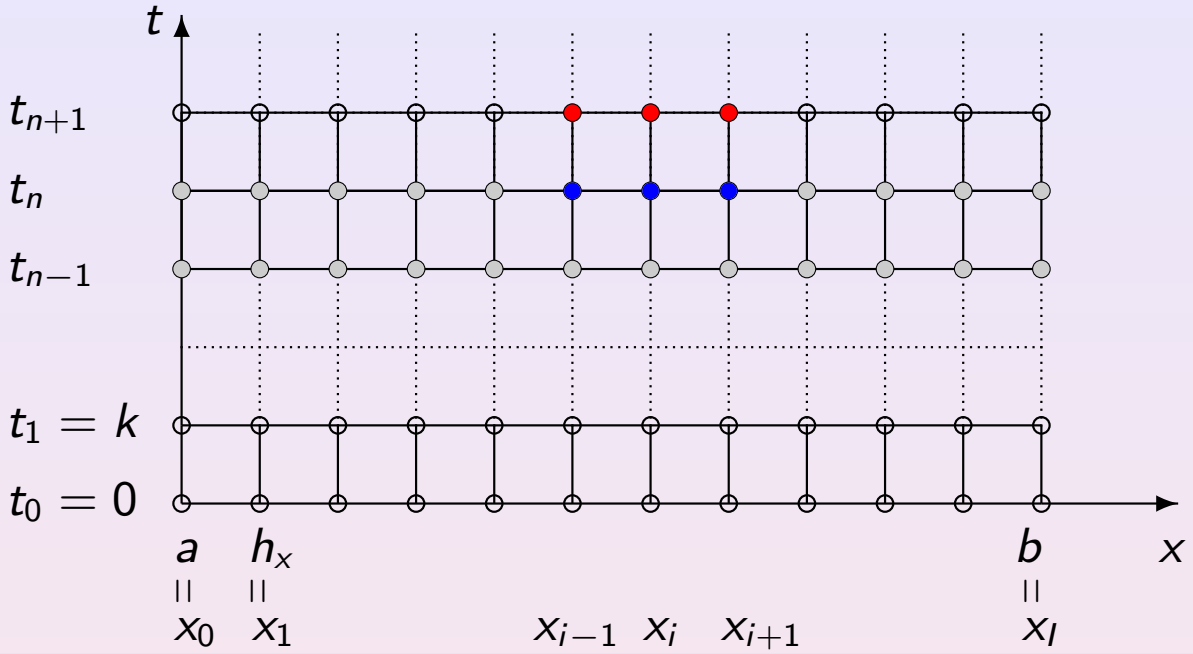
$$U_{ij}^{n+1} - \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n+1} = U_{ij}^{n} + \frac{1}{2} A \left(r_x \, \delta_x^2 + r_y \, \delta_y^2 \right) U_{ij}^{n} + \frac{k}{2} \left(f_{ij}^{n} + f_{ij}^{n+1} \right)$$

$$0 \leq i \leq I, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N.$$

$$r_x = A \frac{k}{h_x^2}, r_y = A \frac{k}{h_y^2}$$

Unconditionally stable and Convergence of second order in time.

ADI schemes (Peaceman-Rachford scheme)



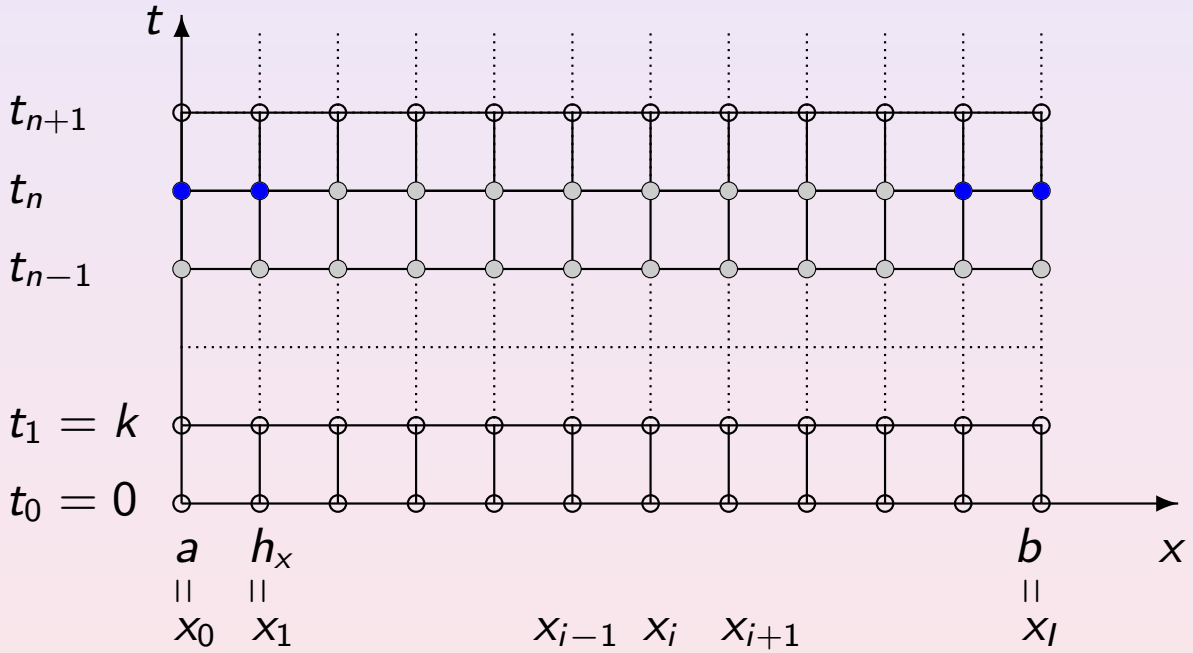
$$
\begin{aligned}
U_{ij}^{n+\frac{1}{2}} - U_{ij}^{n} &= \frac{A}{2}\left(r_x\,\delta_x^2\,U_{ij}^{n+\frac{1}{2}} + r_y\,\delta_y^2\,U_{ij}^{n} + \frac{k}{2}\,f_{ij}^{n}\right. \\
U_{ij}^{n+1} - U_{ij}^{n+\frac{1}{2}} &= \frac{A}{2}\left(r_x\,\delta_x^2\,U_{ij}^{n+\frac{1}{2}} + r_y\,\delta_y^2\,U_{ij}^{n+1} + \frac{k}{2}\,f_{ij}^{n+1}\right.
\end{aligned}
$$

$$
r_x = A\frac{k}{h_x^2},\; r_y = A\frac{k}{h_y^2}
$$

Unconditionally stable and Convergence of second order in time.

Neumann boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = g$$

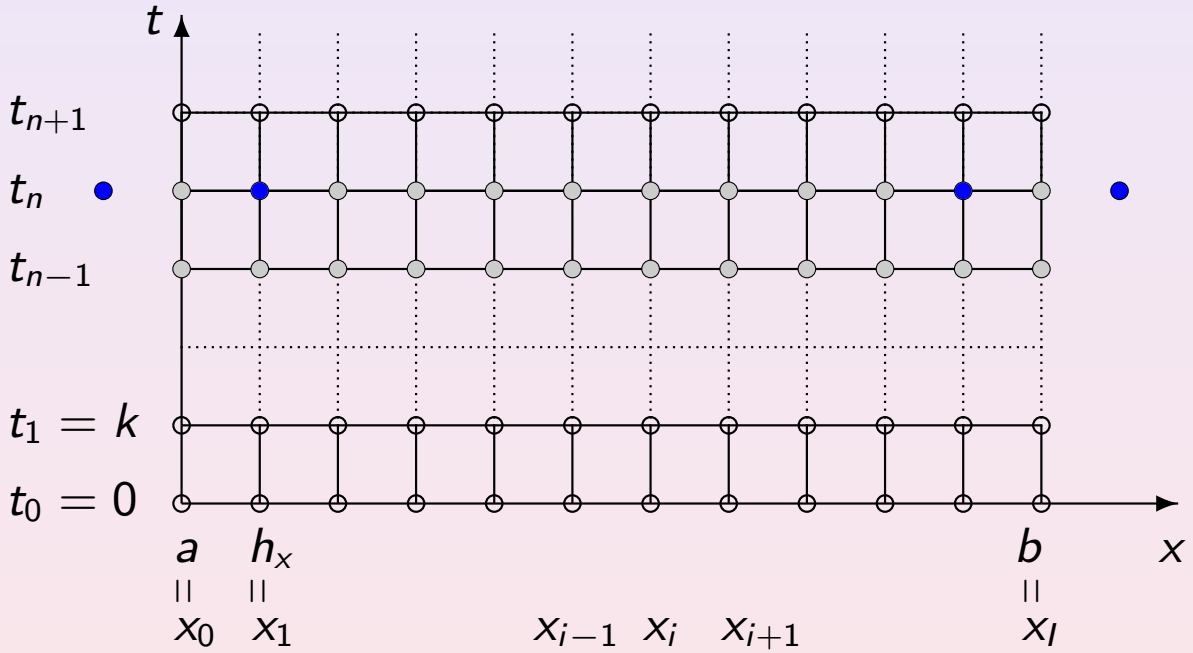


First order method

$$-\left.\frac{\partial u}{\partial x}\right|_{x=x_0} \approx \frac{C_{1j}^n - C_{0j}^n}{h_x} = g_{0j}^n, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=x_I} \approx \frac{C_{I+1,j}^n - C_{Ij}^n}{h_x} = g_{Ij}^n$$

Neumann boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = g$$



Second order method

$$-\left.\frac{\partial u}{\partial x}\right|_{x=x_0} \approx \frac{C_{1j}^n - C_{-1j}^n}{2\,h_x} = g_{0j}^n, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=x_I} \approx \frac{C_{I+1j}^n - C_{I-1,j}^n}{2\,h_x} = g_{Ij}^n$$

References

- J.W. Thomas,
  Numerical Partial Differential Equations. Finite Difference Methods.
  Text in Applied Mathematics. Springer Verlag, 1995
- K.W. Morton, D.F. Mayers,
  Numerical Solution of Partial Differential Equations.
  Cambridge University Press, 1994.

Part II

Age-Structured Population Models

Structured Population Models

- Modeling the interaction of different population communities in demography, epidemiology, ecology, cell kinetics, tumor g...
- Individuals are distinguished by age, size, maturity or some other individual physical characteristic.
- The structure of the population at a given time, and possibly some enviromental input as time evolves, completely determines the dynamical behaviour of the population.
- Mathematical models: initial-boundary value problems for a (system) partial differential equation.
- Independent variables: time, t, and structure variables, $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^k$.
- Dependent variables: the non-negative density functions with respect to these independent variables $u_j(\mathbf{x}, t), j = 1, \ldots, m$.
- The number of individuals at time t of the (sub)populations with individual state variables in $D \subset \mathcal{X}$ is $\int_D u_j(\mathbf{x}, t) d\mathbf{x}, \quad j = 1, \ldots, m$

Structured Population Models

Nonlinear first-order hyperbolic pdes (balance law)

$t > 0, \quad x_0 < x < \infty$, ($x_0 = 0$ age-dependent models)

$$\frac{\partial u}{\partial t} + \frac{\partial (g(x, I_g(t), t)u)}{\partial x} = -\mu(x, I_\mu(t), t)u.$$

- Non-negative function $\mu(x, I_\mu(t), t)$, the mortality rate.
- Non-negative function $g(x, I_g(t), t)$, the growth rate. $g \equiv 1$, age dependent models.
- Weighted averages of the density function u are

$$I_\mu(t) = \int_{x_0}^{\infty} \gamma_\mu(x) u(x,t) dx, I_g(t) = \int_{x_0}^{\infty} \gamma_g(x) u(x,t) dx.$$

- $x_0 > 0$ is the initial magnitude of x at birth.

Structured Population Models

The boundary condition (birth law),

$$g(x_0, I_g(t), t)u(x_0, t) = \int_{x_0}^{\infty} \beta(x, I_\beta(t), t)u(x, t)dx, \quad t \geq 0,$$

- Non-negative function $\beta(x, I_\beta(t), t)$, the fertility rate.
- Weighted average of the density function u is

$$I_\beta(t) = \int_{x_0}^{\infty} \gamma_\beta(x)u(x, t)dx.$$

Initial condition

$$u(x, 0) = \phi(x), \quad x_0 \leq x \leq \infty,$$

- Initial x-specific distribution $\phi(x)$.

Structured Population Models

J.A.J. Metz and E.O. Dieckmann, editors.
The Dynamics of Physiologically Structured Populations, volume 68 of *Springer Lecture Notes in Biomathematics*. Springer, Heildelberg, 1986.

An extensive study of physiologically structured population dynamics, with discussion of the biological background of such models.

Numerical methods.

- Quantitative information from the model.
- Inverse problems in which we must determine some estimations for the growth function or the fertility and mortality rates from the life history of the population.
- Approximate essential parameters.

Outline

2. Bounded Mortality — Infinite age
   - Gurtin-MacCamy models
   - Spurious oscilations
   - Reproductive Cycle of Monogonont Rotifera

Bounded mortality

1. Demography
2. Educational planning: Interest on the prediction of births, school-age children, ...
3. Social planning: Evolution of the medium-age population: individuals past retirement age, ...

Age-Structured population model

$$u_t + u_a = -\mu(a, I_\mu(t), t)\, u, \quad 0 < a < \infty, \quad t > 0,$$

$$u(0,t) = \int_0^\infty \alpha(a, I_\alpha(t), t)\, u(a,t)\, da, \quad t > 0,$$

$$u(a,0) = u_0(a), \quad 0 \le a \le \infty,$$

$$I_\phi(t) = \int_0^\infty \gamma_\phi(a)\, u(a,t)\, da, \quad t \ge 0, \quad \phi = \alpha, \mu.$$

- a, t age and time.
- $u(x,t)$ age-specific density of individuals.
- $\mu(x, I_\mu(t), t)$ age-specific mortality rate function.
- $\beta(x, I_\beta(t), t)$ age-specific fecundity rate function.
- $I_g(t)$, $I_\alpha(t)$, $I_\mu(t)$ weighed populations.
- $u_0(x)$ initial size distribution.

References

- G.F. Webb.
  Theory of Nonlinear Age-Dependent Population Dynamics.
  Marcel Dekker, eds, New York, 1985.
- M. Iannelli.
  Mathematical Theory of Age-Structured Population Dynamics
  Applied Mathematics Monographs. C.N.R., Giardini Editori e Stampatori, Pisa, 1994.
- J.M. Cushing.
  An Introduction to Structured Population Dynamics.
  CMB-NSF Regional Conference Series in Applied Mathematics. SIAM, 1998.

Numerical Schemes

L. M. Abia, O. Angulo and J. C. López-Marcos,
Age-structured population dynamics models and their numerical solutions,
Ecol. Model. **188** (2005) 112-136.

Classification

- Finite difference methods
- Characteristics curves
  - Discretize de set of ODEs
  - Representation of the solution

Finite Difference Methods

- A is the maximum age.
- T is the final integration time.
- $J \in \mathbb{N}$, $x_j = j\,h$, $0 \leq j \leq J$, $h = \dfrac{A}{J}$,

  $t_n = n\,k$, $0 \leq n \leq N$, $N = \left[\dfrac{T}{k}\right]$.
- Discretization of

$$u_t + u_a = -\mu(a, I_\mu(t), t)\, u.$$

- Boundary condition discretization

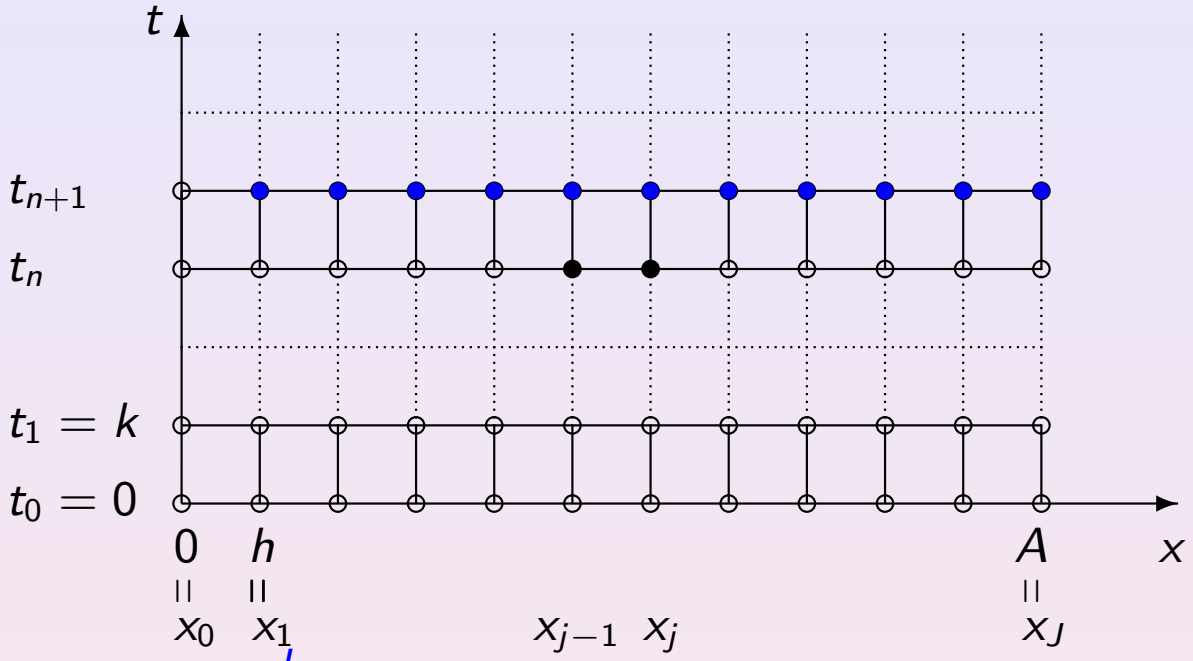
$$u(0,t) = \int_0^A \alpha(a, I_\alpha(t), t)\, u(a,t)\, da.$$

- Initial condition on the grid.

References

- J.C. López-Marcos.
  Upwind scheme (1991).
- G. Fairweather and J.C. López-Marcos.
  Box method (1991).
- D. Sulsky.
  Lax-Wendroff method (1993).
- G. Fairweather and J.C. López-Marcos.
  Explicit extrapolated box scheme (1994).
- M.Y. Kim and E.J. Park.
  Extrapolated upwind scheme (1995).

Upwind method



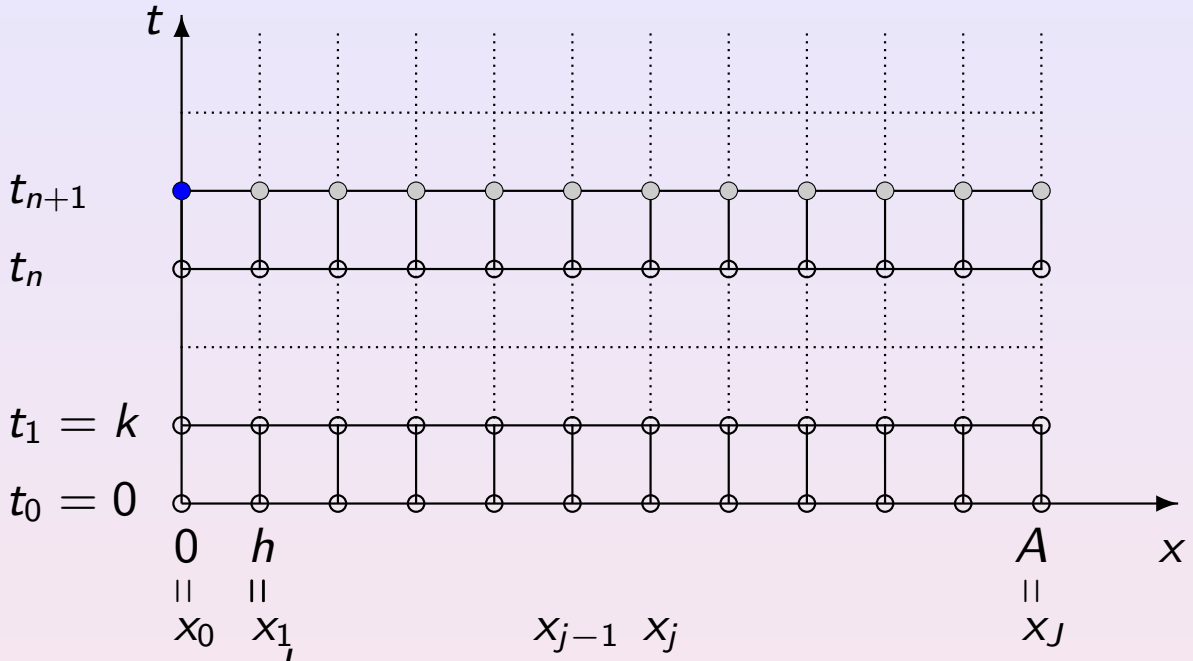
- $U_j^{n+1} = U_j^n + \frac{k}{h}\left(U_j^n - U_{j-1}^n\right) - \mu(x_j, Q_h(\gamma_\mu \mathbf{U}^n), t_n)\, U_j^n, \qquad j = 1, 2, \ldots, J$
- $U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\, \mathbf{U}^{n+1})$.

Notation

$$Q_h(\mathbf{V}^n) = \sum_{j=1}^{J} h\, V_j$$

$$\alpha_j^n(\mathbf{U}) = \alpha\left(x_j, Q_h(\gamma_\alpha \mathbf{U}^n), t_n\right), \; j = 0, 1, \ldots, J.$$

Upwind method



- $U_j^{n+1} = U_j^n + \frac{k}{h}\left(U_j^n - U_{j-1}^n\right) - \mu(x_j, Q_h(\gamma_\mu \mathbf{U}^n), t_n)\, U_j^n, \qquad j = 1, 2, \ldots, J$
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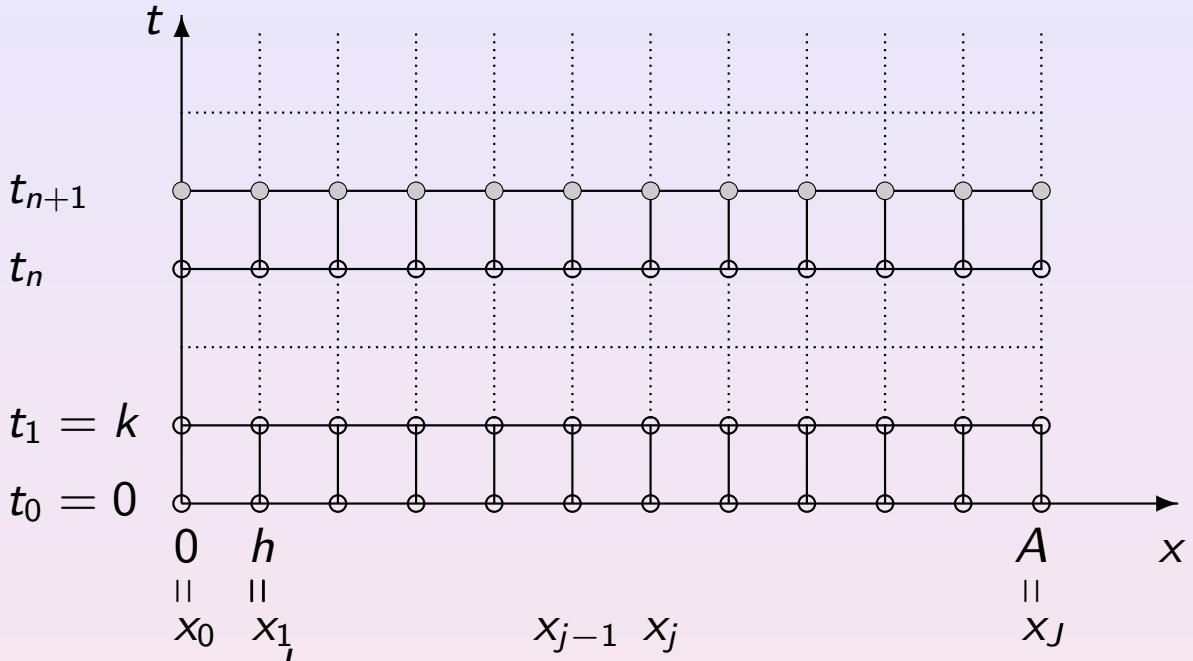
Upwind method



- $U_j^{n+1} = U_j^n + \dfrac{k}{h}\left(U_j^n - U_{j-1}^n\right) - \mu(x_j, Q_h(\gamma_\mu \mathbf{U}^n), t_n)\, U_j^n, \qquad j = 1, 2, \ldots, J$
- $U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\, \mathbf{U}^{n+1})$.

Notation

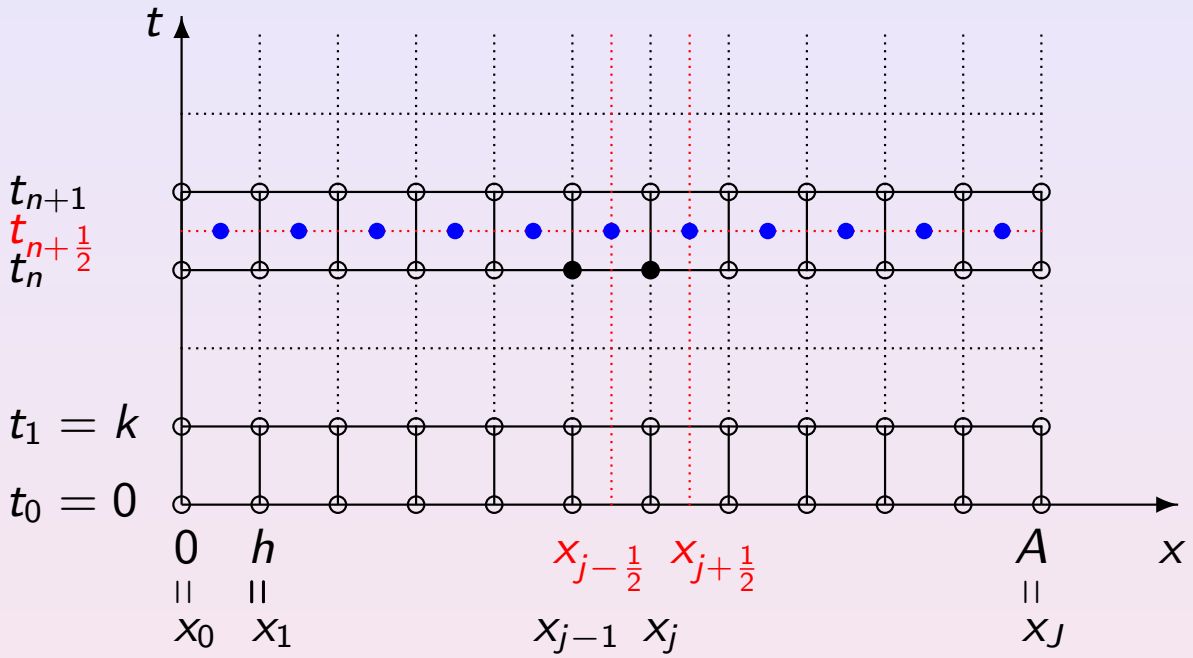
$$Q_h(\mathbf{V}^n) = \sum_{j=1}^{J} h\, V_j$$

$$\alpha_j^n(\mathbf{U}) = \alpha\left(x_j, Q_h(\gamma_\alpha \mathbf{U}^n), t_n\right),\ j = 0, 1, \ldots, J.$$

Extrapolated Upwind method

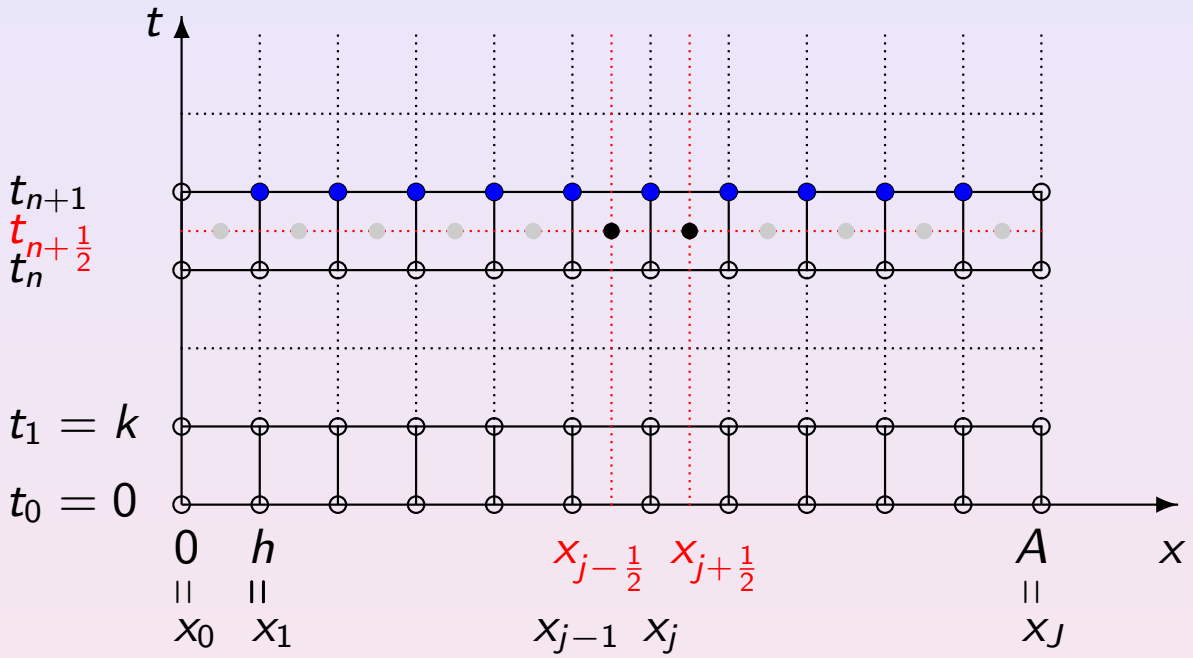
$$U_j^n = \frac{U_j^{n-1} + \frac{k}{h}\left(U_j^{n-1} - U_{j-1}^{n-1}\right)}{1 + k\,\mu(x_j, Q_h(\gamma_\mu\,\mathbf{U}^{n-1}), t_{n-1})}, \quad j = 1, 2, \ldots, J$$

Lax-Wendroff method



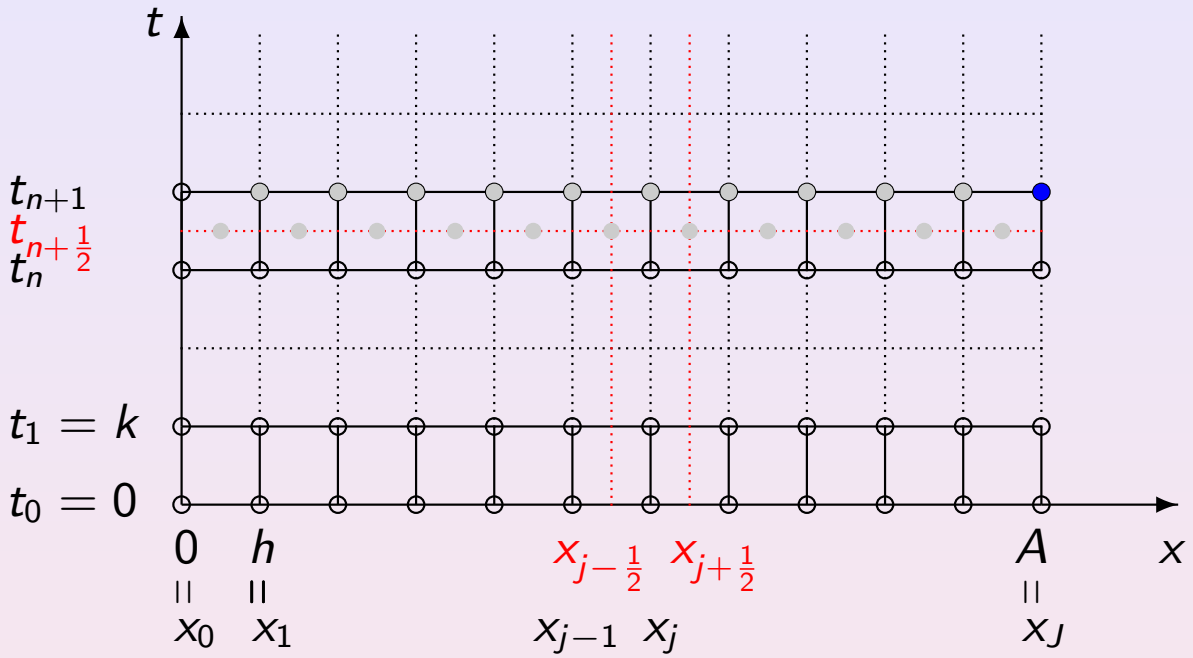
- $U^{n+\frac{1}{2}}_{j-\frac{1}{2}} = U^n_{j-\frac{1}{2}} - \frac{k}{2\,h}\left(U^n_j - U^n_{j-1}\right) - \frac{k}{2}\mu^n_{j-\frac{1}{2}}(\mathbf{U})\,U^n_{j-\frac{1}{2}}, \quad j = 1, 2, \ldots, J$
- $U^{n+1}_j = U^n_j - \frac{k}{h}\left(U^{n+\frac{1}{2}}_{j+\frac{1}{2}} - U^{n+\frac{1}{2}}_{j-\frac{1}{2}}\right) - k\,\mu^{n+\frac{1}{2}}_j(\mathbf{U})\,U^{n+\frac{1}{2}}_j, \quad j = 1, 2, \ldots, J-1.$
- $U^{n+1}_J = 0$
- $U^{n+1}_0 = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}).$

Lax-Wendroff method



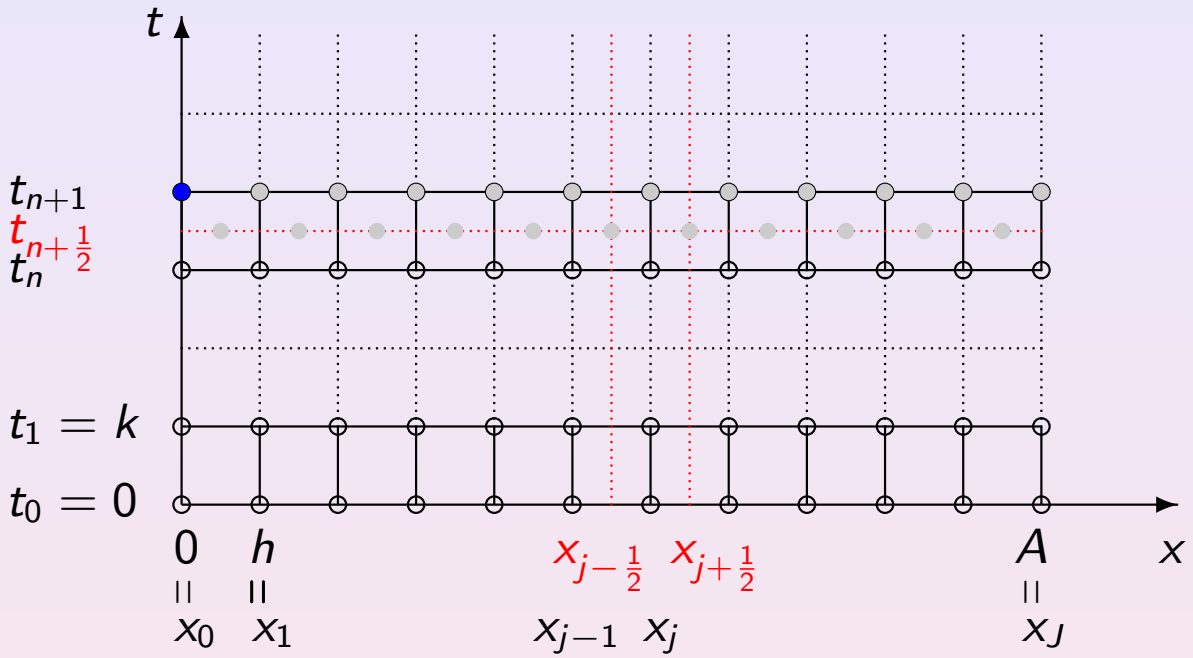
- $U_{j-\frac{1}{2}}^{n+\frac{1}{2}} = U_{j-\frac{1}{2}}^{n} - \frac{k}{2\,h}\left(U_j^n - U_{j-1}^n\right) - \frac{k}{2}\mu_{j-\frac{1}{2}}^n(\mathbf{U})\,U_{j-\frac{1}{2}}^n, \quad j = 1, 2, \ldots, J$
- $U_j^{n+1} = U_j^n - \frac{k}{h}\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) - k\,\mu_j^{n+\frac{1}{2}}(\mathbf{U})\,U_j^{n+\frac{1}{2}}, \quad j = 1, 2, \ldots, J-1.$
- $U_J^{n+1} = 0$
- $U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}).$

Lax-Wendroff method



- $U_{j-\frac{1}{2}}^{n+\frac{1}{2}} = U_{j-\frac{1}{2}}^{n} - \frac{k}{2\,h}\left(U_j^n - U_{j-1}^n\right) - \frac{k}{2}\mu_{j-\frac{1}{2}}^{n}(\mathbf{U})\,U_{j-\frac{1}{2}}^{n}, \quad j = 1, 2, \ldots, J$
- $U_j^{n+1} = U_j^n - \frac{k}{h}\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) - k\,\mu_j^{n+\frac{1}{2}}(\mathbf{U})\,U_j^{n+\frac{1}{2}}, \quad j = 1, 2, \ldots, J-1.$
- $U_J^{n+1} = 0$
- $U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}).$

Lax-Wendroff method



- $U^{n+\frac{1}{2}}_{j-\frac{1}{2}} = U^n_{j-\frac{1}{2}} - \frac{k}{2\,h}\left(U^n_j - U^n_{j-1}\right) - \frac{k}{2}\mu^n_{j-\frac{1}{2}}(\mathbf{U})\,U^n_{j-\frac{1}{2}}, \quad j = 1, 2, \ldots, J$
- $U^{n+1}_j = U^n_j - \frac{k}{h}\left(U^{n+\frac{1}{2}}_{j+\frac{1}{2}} - U^{n+\frac{1}{2}}_{j-\frac{1}{2}}\right) - k\,\mu^{n+\frac{1}{2}}_j(\mathbf{U})\,U^{n+\frac{1}{2}}_j, \quad j = 1, 2, \ldots, J-1.$
- $U^{n+1}_J = 0$
- $U^{n+1}_0 = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}).$

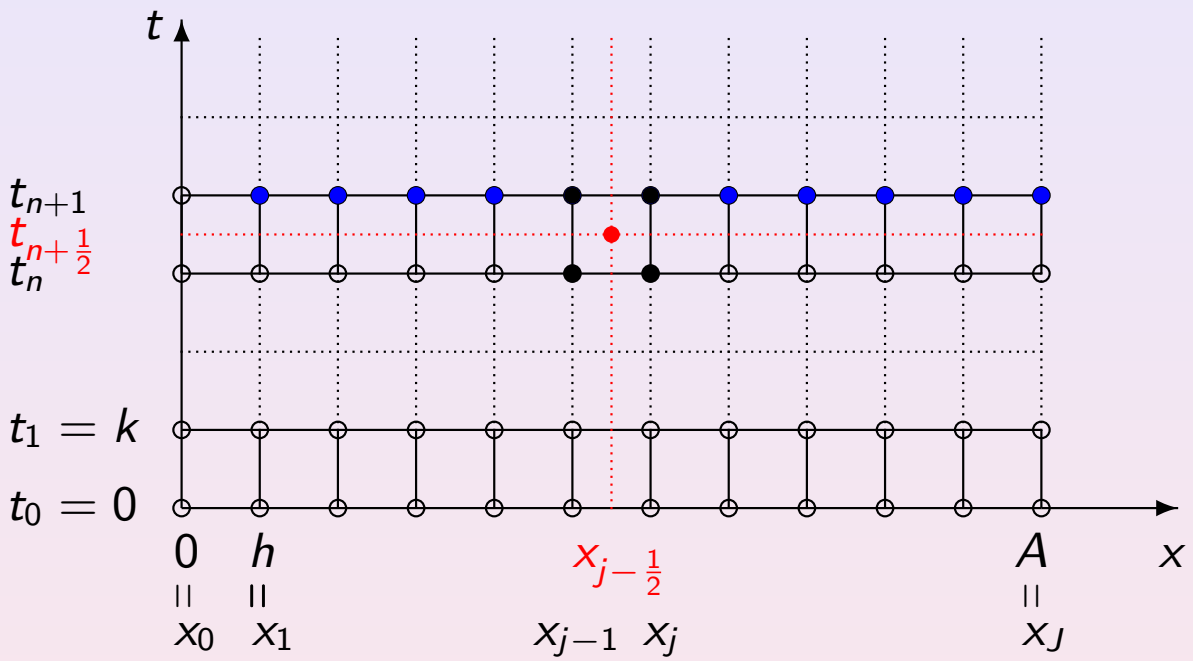
Lax-Wendroff method. Notation.

$$\mu^n_{j-\frac{1}{2}}(\mathbf{U}) = \mu\left(x_{j-\frac{1}{2}}, Q_h\left(\gamma_\mu \mathbf{U}^n\right), t_n\right),\ j = 1, 2, \ldots, J,$$
$$x_{j-\frac{1}{2}} = \frac{1}{2}\left(x_{j-1} + x_j\right),\ U^n_{j-\frac{1}{2}} = \frac{1}{2}\left(U^n_{j-1} + U^n_j\right),\ j = 1, 2, \ldots, J;$$
$$Q_h(\gamma_s \mathbf{U}^n) = \sum_{j=1}^{J} \frac{h}{2}\left(\gamma_s(x_{j-1})\, U^n_{j-1} + \gamma_s(x_j)\, U^n_j\right),\ s = \mu, \alpha.$$

$$U_j^{n+\frac{1}{2}} = \frac{1}{2}\left(U^{n+\frac{1}{2}}_{j+\frac{1}{2}} + U^{n+\frac{1}{2}}_{j-\frac{1}{2}}\right),$$
$$\mu_j^{n+\frac{1}{2}}(\mathbf{U}) = \mu\left(x_j, Q_h\left(\gamma_\mu \mathbf{U}^{n+\frac{1}{2}}\right), t_n + \frac{k}{2}\right),\ j = 1, 2, \ldots, J-1;$$
$$Q_h(\gamma_\mu \mathbf{U}^{n+\frac{1}{2}}) = \sum_{j=0}^{J-1} h\,\gamma_\mu(x_{j+\frac{1}{2}})\, U^{n+\frac{1}{2}}_{j+\frac{1}{2}},$$

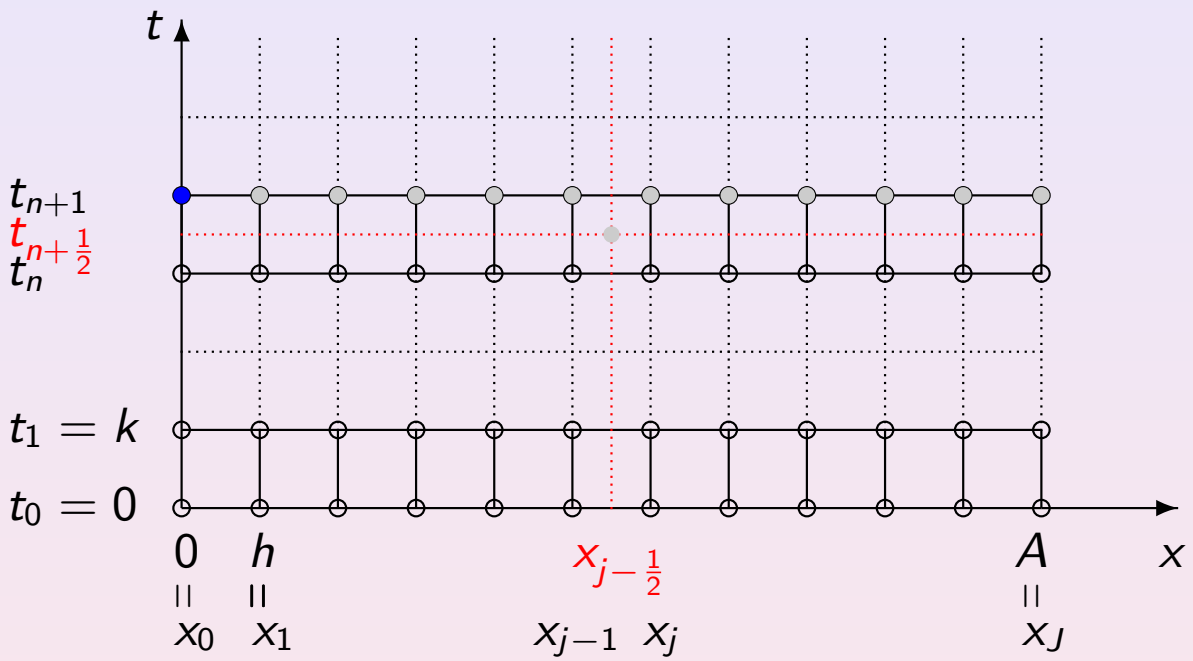
$$\alpha^n_j(\mathbf{U}) = \alpha\left(x_j, Q_h(\gamma_\alpha \mathbf{U}^n), t_n\right),\ j = 0, 1, \ldots, J.$$

Box method



- $$\frac{D\,U_j^n + D\,U_{j-1}^n}{2\,k} + \frac{U_j^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}}}{h} = -\mu_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\mathbf{U})\; U_{j-\frac{1}{2}}^{n+\frac{1}{2}}, \quad 1 \le j \le J,$$
- $U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1})$.

Box method



- $$\frac{D\,U_j^n + D\,U_{j-1}^n}{2\,k} + \frac{U_j^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}}}{h} = -\mu_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\mathbf{U})\; U_{j-\frac{1}{2}}^{n+\frac{1}{2}}, \quad 1 \le j \le J,$$
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Box method. Notation.

$$D\,U_j^n := U_j^{n+1} - U_j^n,\ U_j^{n+\frac{1}{2}} := \frac{1}{2}\left(U_j^{n+1} + U_j^n\right),\ 0 \le j \le J,$$
$$x_{j-\frac{1}{2}} = \frac{1}{2}\left(x_{j-1} + x_j\right),\ U_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{j-1}^{n+\frac{1}{2}} + U_j^{n+\frac{1}{2}}\right),\ 1 \le j \le J,$$
$$\mu_{j-\frac{1}{2}}^{n+\frac{1}{2}}\left(\mathbf{U}\right) = \mu\left(x_{j-\frac{1}{2}}, Q_h\left(\gamma_\mu\,\mathbf{U}^{n+\frac{1}{2}}\right), t_n + \frac{k}{2}\right),\ 1 \le j \le J,$$

$$U_{j-\frac{1}{2}}^n = \frac{1}{2}\left(U_{j-1}^n + U_j^n\right),$$
$$Q_h\left(\gamma_s\,\mathbf{U}^{n-\frac{1}{2}}\right) = \sum_{j=1}^{J} h\,\gamma_\mu\left(x_{j-\frac{1}{2}}\right)\,U_{j-\frac{1}{2}}^{n-\frac{1}{2}},\ 1 \le n \le N,\ s = g, \mu.$$

$$\alpha_j^n\left(\mathbf{U}\right) = \alpha\left(x_j, Q_h\left(\gamma_\alpha\,\mathbf{U}^n\right), t_n\right),\ Q_h\left(\gamma_\alpha\,\mathbf{U}^n\right) = \sum_{j=1}^{J} h\,\gamma_\alpha\left(x_{j-\frac{1}{2}}\right)\,U_{j-\frac{1}{2}}^n,$$
$$0 \le n \le N.$$

Extrapolated box method

$$\mu_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\mathbf{U}) = \mu\left(x_{j-\frac{1}{2}}, Q_h\left(\gamma_\mu \frac{3\,\mathbf{U}^n - \mathbf{U}^{n-1}}{2}\right), t_n + \frac{k}{2}\right), \; 1 \leq j \leq J,$$

$$\alpha_j^n(\mathbf{U}) = \alpha\left(x_j, Q_h\left(\gamma_\alpha\left(2\,\mathbf{U}^{n-1} - \mathbf{U}^{n-2}\right)\right), t_n\right),$$

Integration along characteristics.

$$u_t(a,t) + u_a(a,t) = -\mu(a, I_\mu(t), t)\, u(a,t), \quad 0 < x < \infty.$$

- $x(t; t_0, a_0) = t + t_0 - a_0$,
- $w(t; t_0, x_0) = u(x(t; t_0, x_0), t),\ t \geq t_0$.

Integration along characteristics.

$$u_t(a,t) + u_a(a,t) = -\mu(a, I_\mu(t), t)\ u(a,t), \quad 0 < x < \infty.$$

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System of Ordinary Differential Equations

$$\begin{cases} \dfrac{d}{dt} w(t; t_0, a_0) & = & -\mu\left(x\left(t; t_0, a_0\right), I_\mu(t), t\right)\ w(t; t_0, a_0), \quad t \geq t_0, \\ w(t_0; t_0, a_0) & = & u(a_0, t_0), \end{cases}$$

Theoretical Representation of the solution

$$w(t; t_0, a_0) = u(a_0, t_0)\ \exp\left\{-\int_{t_0}^{t} \mu\left(x\left(\tau; t_0, a_0\right), I_\mu(\tau), \tau\right)\ d\tau\right\}, \quad t \geq t_0.$$

Characteristics methods
Discretize the set of ODEs

- A is the maximum age.
- T is the integration final time.
- $J \in \mathbb{N}$, $X_j = j\,h$, $0 \leq j \leq J$, $h = \dfrac{A}{J}$,

  $t_n = n\,k$, $0 \leq n \leq N$, $N = \left[\dfrac{T}{h}\right]$.
- Discretization of

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt} w(t; t_n, X_j) & = & -\mu\left(x\left(t; t_n, X_j\right), l_\mu(t), t\right)\ w(t; t_n, X_j), \quad t \geq t_0, \\ w(t_0; t_n, X_j) & = & u(X_j, t_n), \end{array} \right.$$

- Boundary condition discretization

$$u(0, t) = \int_0^A \alpha(a, l_\alpha(t), t)\, u(a, t)\, da,$$

- Initial condition on the grid.

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  First order accuracy (1987).
- R. Kannan and R. Ortega.
  First order accuracy (1989).
- T.V. Kostova.
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  Second Order accuracy (1993).
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  General Order accuracy (1995).

Characteristics methods
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$$w(t; t_n, X_j) = u(X_j, t_n)\, \exp\left\{ - \int_{t_n}^{t} \mu\left(x\left(\tau; t_n, X_j\right), I_\mu(\tau), \tau\right)\, d\tau \right\}.$$

- Boundary condition discretization

$$u(0,t) = \int_0^A \alpha(a, I_\alpha(t), t)\, u(a,t)\, da,$$

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L.M. Abia and J.C. López-Marcos.
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Convergence Analysis: Consistency
Upwind method

Truncation error τ_j^{n+1} at (x_j, t_{n+1}):

$$\tau_j^{n+1} := \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} + \mu u_j^n.$$

If u is sufficiently smooth,

$$\tau_j^{n+1} = \frac{\Delta t}{2} u_{tt}(x_j, t_n + \theta_1 \Delta t) - \frac{\Delta x}{2} u_{xx}(x_j - \theta_2 \Delta x, t_n).$$

Consistent of first order accuracy in time and of first order accuracy in age.

Convergence Analysis: Convergence Upwind method

Global error e_j^{n+1} of the scheme, at (x_j, t_{n+1}):

$$e_j^n = U_j^n - u_j^n.$$

$$e_j^{n+1} = (1-\nu)\, e_j^n + \nu\, e_{j-1}^n - \Delta t\, \mu\, e_j^n - \Delta t\, \tau_j^{n+1},$$

If $0 < \nu \leq 1$, $|e_j^{n+1}| \leq (1-\nu)\, |e_j^n| + \nu\, |e_{j-1}^n| + \Delta t\, \mu\, |e_j^n| + \Delta t\, |\tau_j^{n+1}|$.

$$E^n := \max_{j=1,\ldots,J} |e_j^n|,$$

$$E^{n+1} \leq (1 + \Delta t\, \mu)\, E^n + \Delta t\, (\max_{j=1,\ldots,J} |\tau_j^{n+1}|).$$

If $|\tau_j^n| \leq M_\tau$, for all j and n in the domain.

$$E^n \leq \exp(\mu\, n\, \Delta t) E^0 + \frac{1}{\mu} \exp(\mu\, n\, \Delta t)\, M_\tau, \quad 0 \leq n\, \Delta t \leq T.$$

The upwind scheme has first order of Convergence

Experimental Convergence and Efficiency Study

Theoretical Test Problem

$$\mu(x,z,t) = z$$

$$\alpha(x,z,t) = \frac{4\,x\,z\,e^{-x}\,(2-2\,e^{-A}+e^{-t})^2}{(1+z)^2\,(1-e^{-A})\,(1-(1+2A)\,e^{-2A})\,(1-e^{-A}+e^{-t})}\,.$$

$$\gamma_\mu \equiv \gamma_\alpha \equiv 1$$

$$u(x,t) \;=\; \frac{\exp(-x)}{1\,-\,\exp(-A)\,+\,\exp(-t)}\,.$$

Numerical Methods

- Box method
- Characteristics Method based on Runge-Kutta, Abia and López-Marcos (1995)

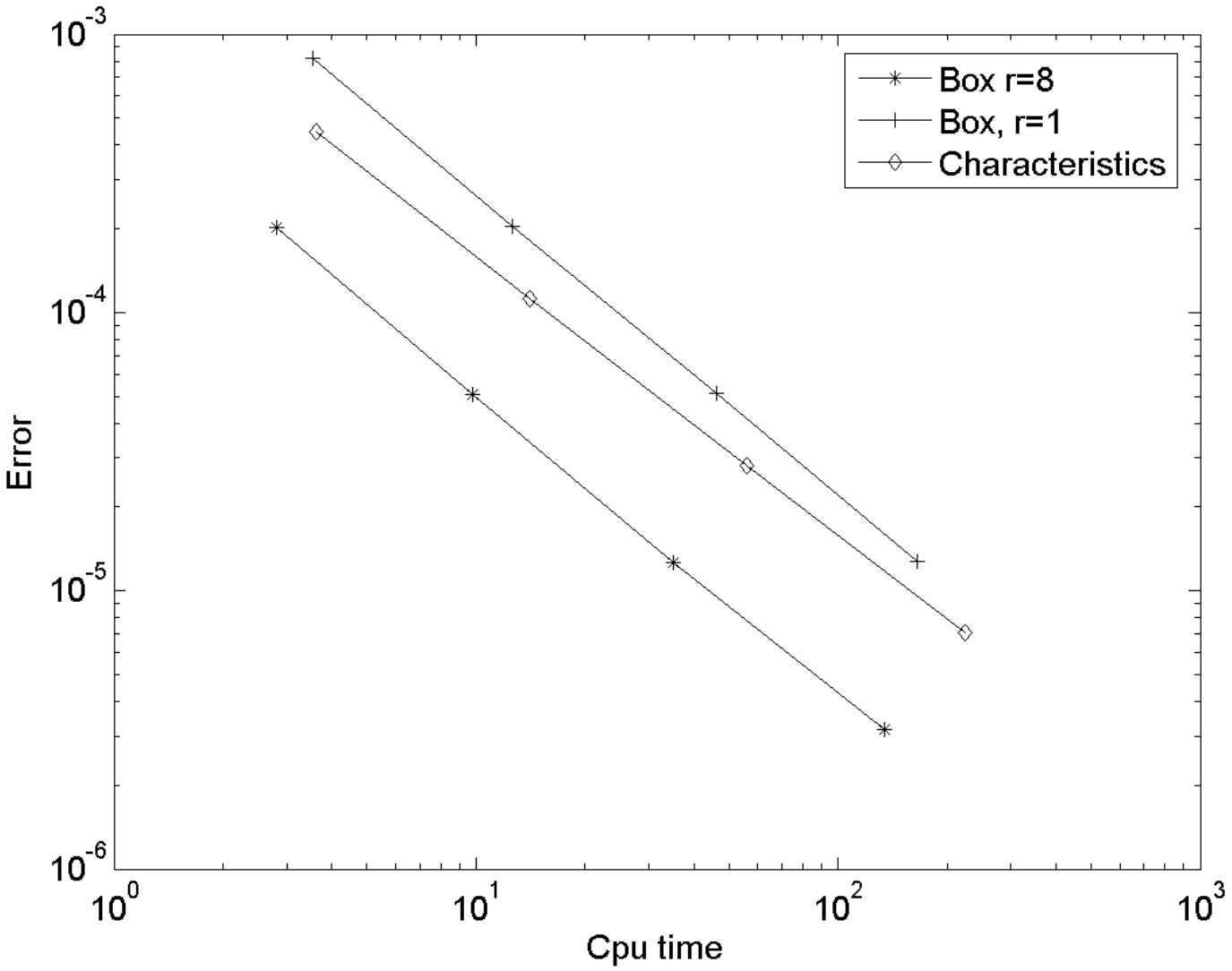
Experimental Convergence - Box Method

$k \backslash h$	3.125E-02	1.563E-02	7.813E-03	3.906E-03	1.953E-03
3.125E-2	8.170E-04	2.029E-04	5.064E-05	1.266E-05	5.905E-06
	| 3.540	| 8.640	| 17.84	| 35.07	| 70.06
1.563E-2	8.192E-04	2.044E-04	5.073E-05	1.266E-05	3.165E-06
	| 7.780	1.999 | 12.57	2.000 | 31.55	2.000 | 65.96	2.000 | 134.7
7.813E-3	8.198E-04	2.049E-04	5.110E-05	1.268E-05	3.165E-06
	| 14.93	1.999 | 27.93	2.000 | 46.07	2.000 | 123.1	2.000 | 251.3
3.906E-3	8.200E-04	2.051E-04	5.124E-05	1.278E-05	3.171E-06
	| 27.98	1.999 | 53.22	2.000 | 104.3	2.000 | 165.1	2.000 | 475.5
1.953E-3	8.201E-04	2.051E-04	5.128E-05	1.281E-05	3.194E-06
	| 51.87	1.999 | 102.0	2.000 | 203.0	2.000 | 395.4	2.000 | 630.4

Experimental Convergence - Characteristcs Scheme

k	Error	Order	cpu-time
3.125E-2	4.445E-4		3.62
1.563E-2	1.120E-4	1.988	14.08
7.813E-3	2.811E-5	1.994	56.17
3.906E-3	7.042E-6	1.997	223.88
1.953E-3	1.762E-6	1.999	883.44

Efficiency


Box r=8
Box, r=1
Characteristics
Error
Cpu time
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{0}
10^{1}
10^{2}
10^{3}

Spurious oscillations
Compatibility conditions

First compatibility condition: continuity

$$u_0(0) = \int_0^A \alpha\left(a, I_\alpha(0), 0\right)\, u_0(a)\, da.$$

$u^0(a) = \exp(-a)$

Box

$k \backslash h$	5.000E-01	2.500E-01	1.250E-01	6.250E-02	3.125E-02
5.000E-1	1.273E-1	3.889E-2	5.049E-2	4.635E-2	4.669E-2
		1.711E+0	-2.683E-1	-3.190E-1	-4.502E-1
2.500E-1	1.273E-1	3.689E-2	4.887E-2	5.737E-2	5.426E-2
		1.786E+0	-3.297E-1	-1.845E-1	-2.273E-1
1.250E-1	1.273E-1	5.441E-2	2.461E-2	5.625E-2	6.170E-2
		1.226E+0	5.841E-1	-2.028E-1	-1.050E-1
6.250E-2	1.273E-1	7.990E-2	6.917E-2	2.648E-2	6.084E-2
		6.715E-1	-3.462E-1	-1.058E-1	-1.133E-1
3.125E-2	1.273E-1	9.915E-2	9.275E-2	7.540E-2	2.717E-2
		3.601E-1	-2.151E-1	-1.244E-1	-3.685E-2

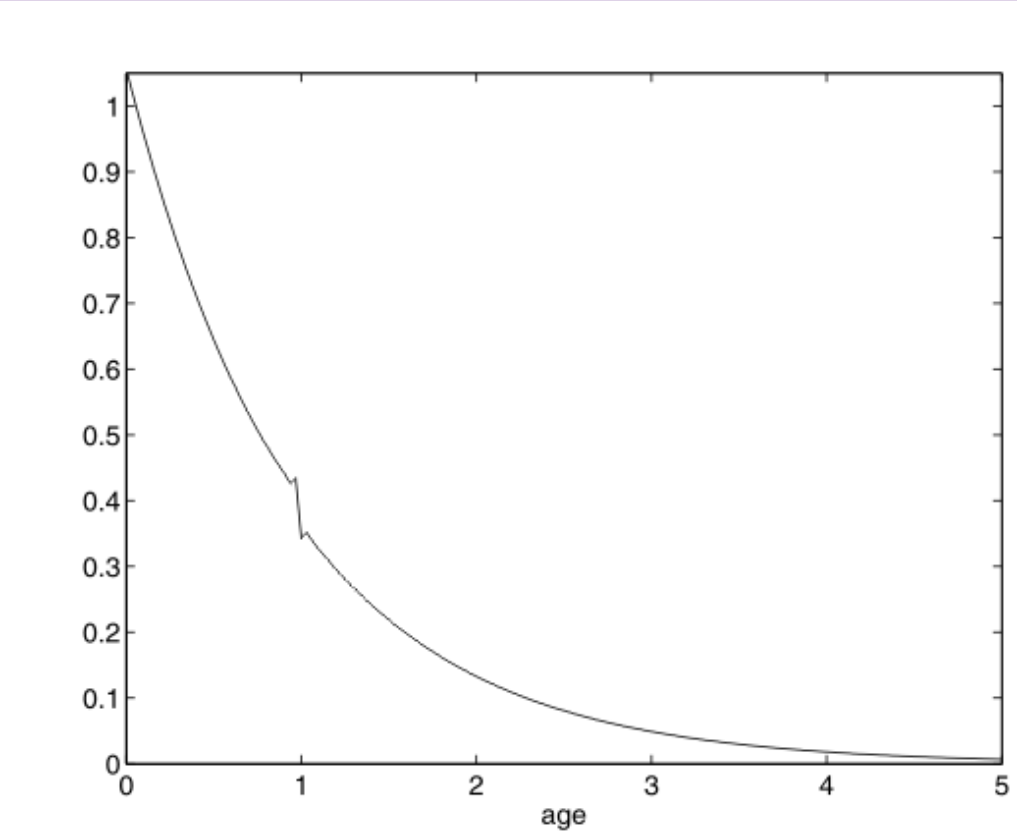
$u^0(a) = \exp(-a)$

Characteristics

k	Error	Orden
5.000E-1	2.476E-1	
2.500E-1	1.499E-2	4.046
1.250E-1	5.253E-3	1.513
6.250E-2	1.915E-3	1.456
3.125E-2	7.538E-4	1.345

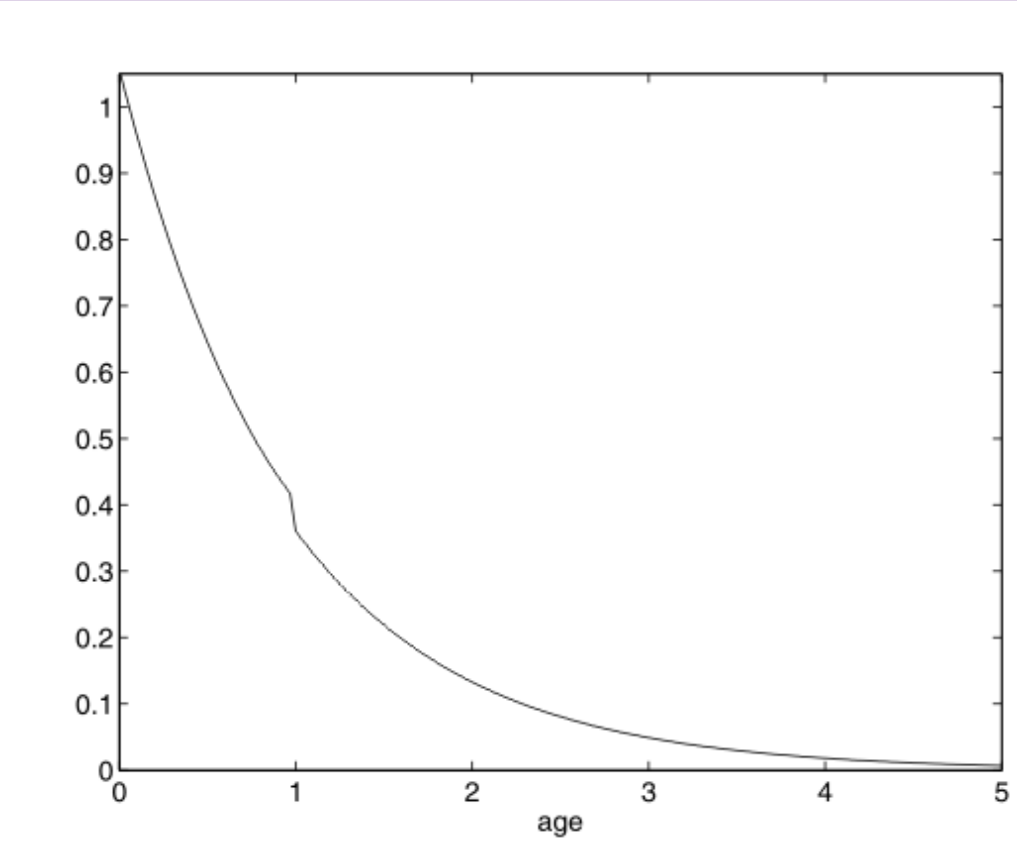
$u^0(a) = \exp(-a)$

box method, $r = 1$, $k = 0.03125$.



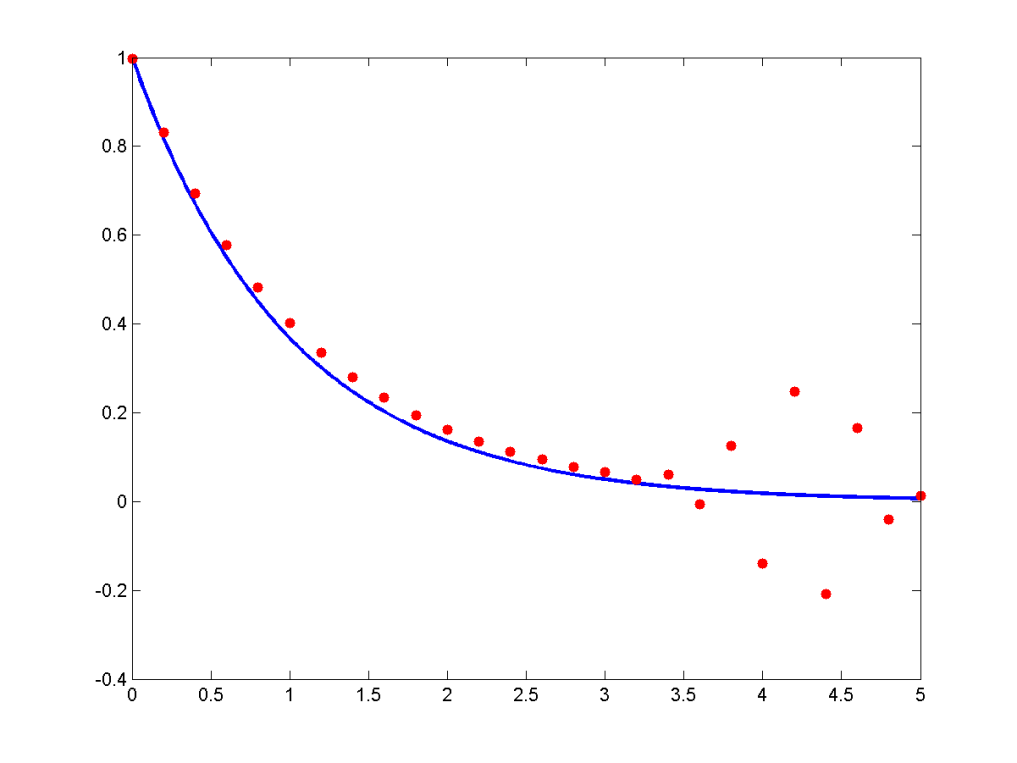
$u^0(a) = \exp(-a)$

Characteristic scheme, $k = 0.03125$



Upwind method

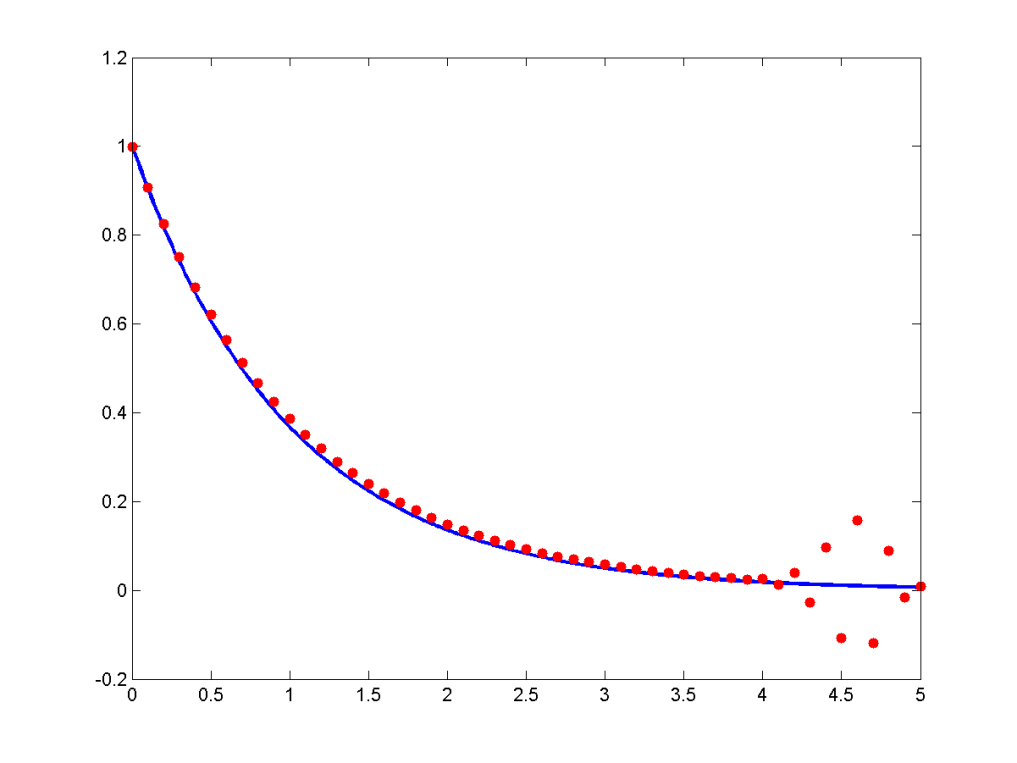
$A = 5,\ T = 5,\ r = 1,\ k = h = 0.2.$



Another

Upwind method

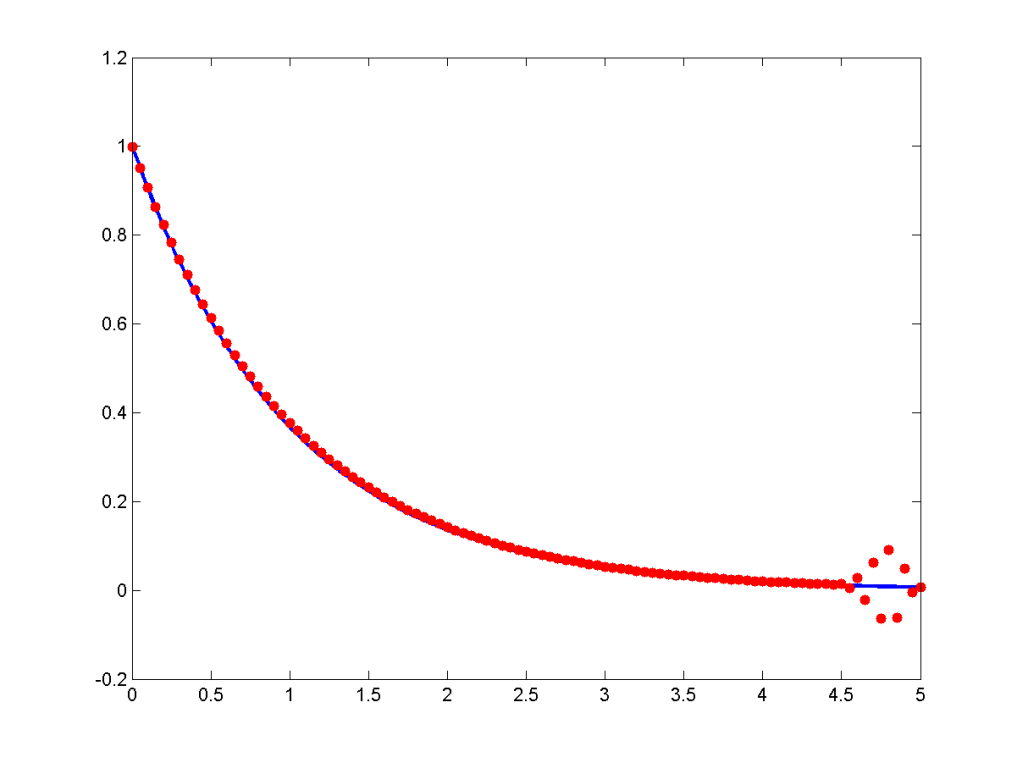
$A = 5,\ T = 5,\ r = 1,\ k = h = 0.1.$



Another

Upwind method

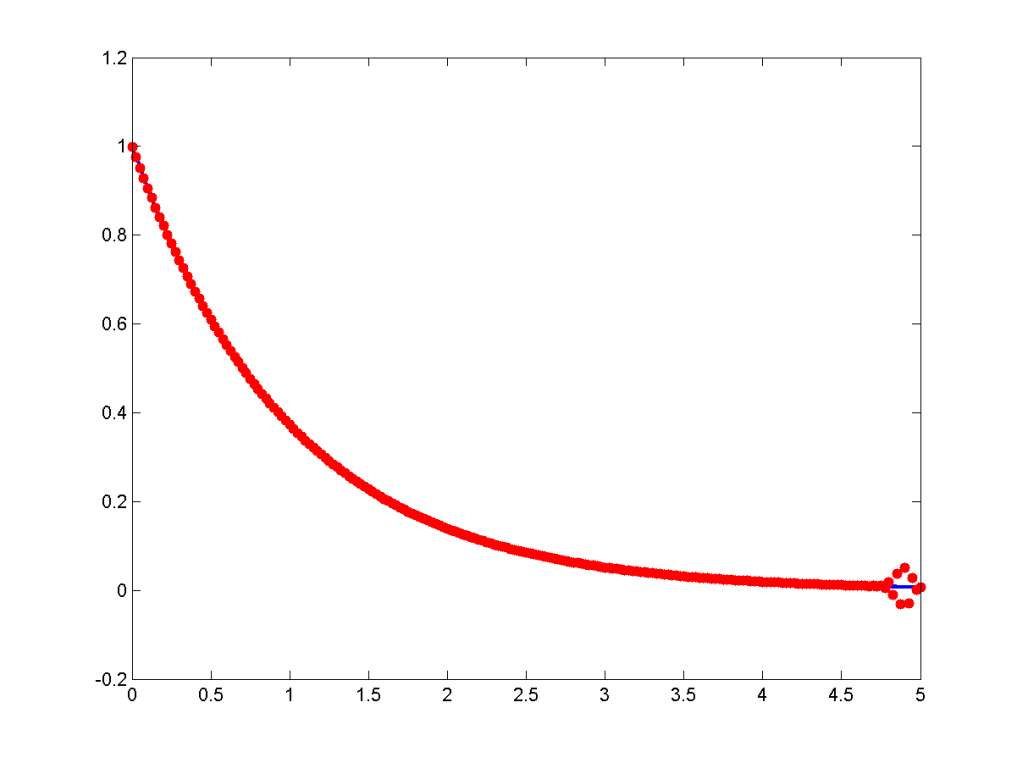
$A = 5,\ T = 5,\ r = 1,\ k = h = 0.05.$



Another

Upwind method

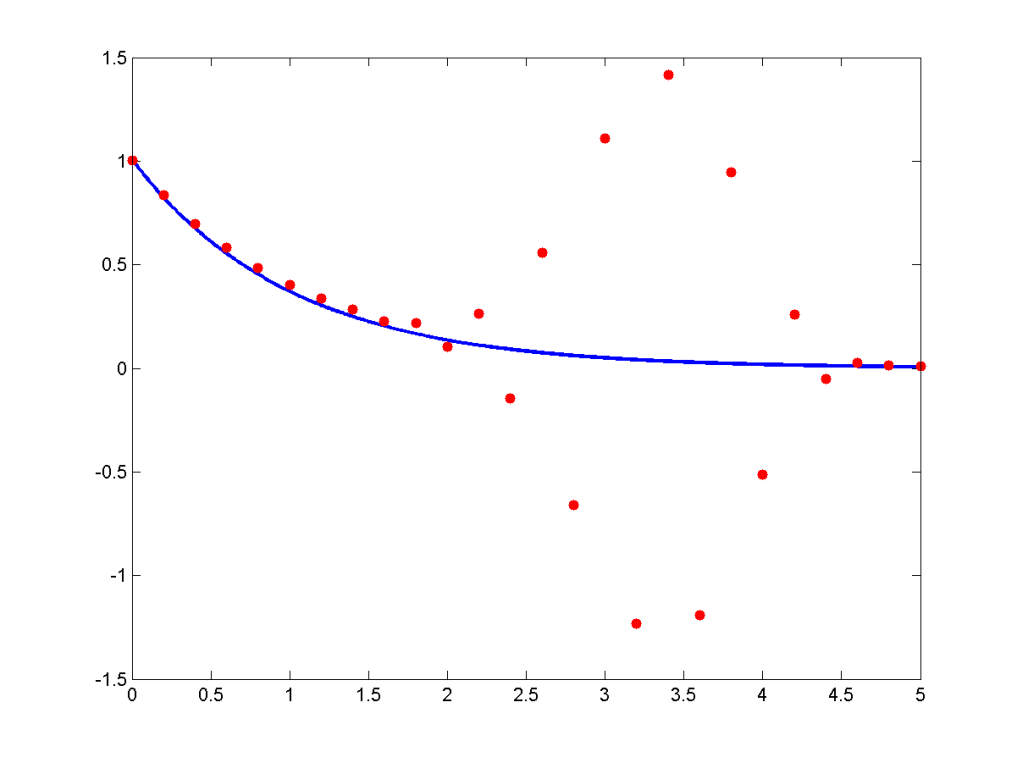
$A = 5,\ T = 5,\ r = 1,\ k = h = 0.025.$



Another

Upwind method

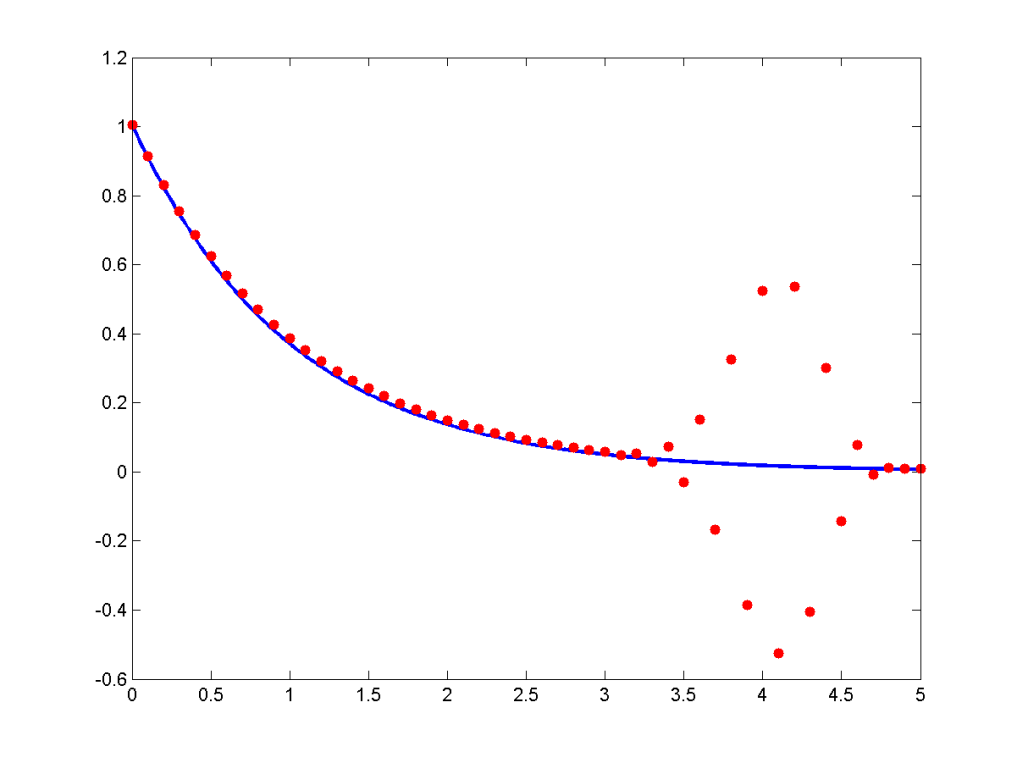
$A = 5,\ T = 10,\ r = 1,\ k = h = 0.2.$



Another

Upwind method

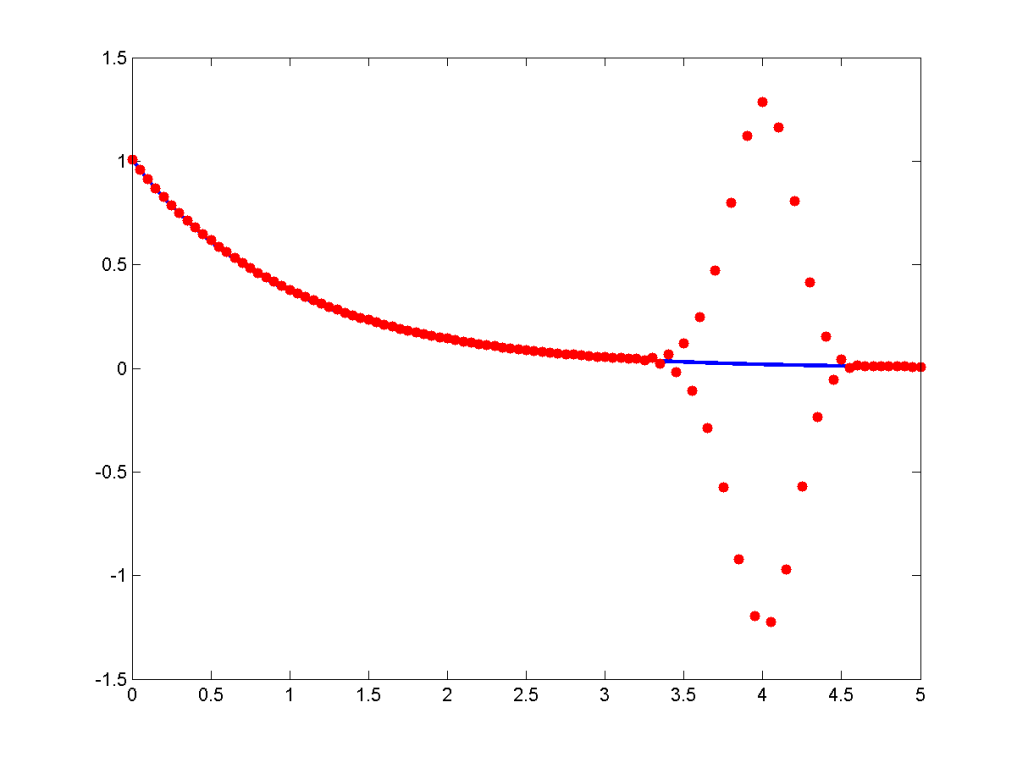
$A = 5,\ T = 10,\ r = 1,\ k = h = 0.1.$



Another

Upwind method

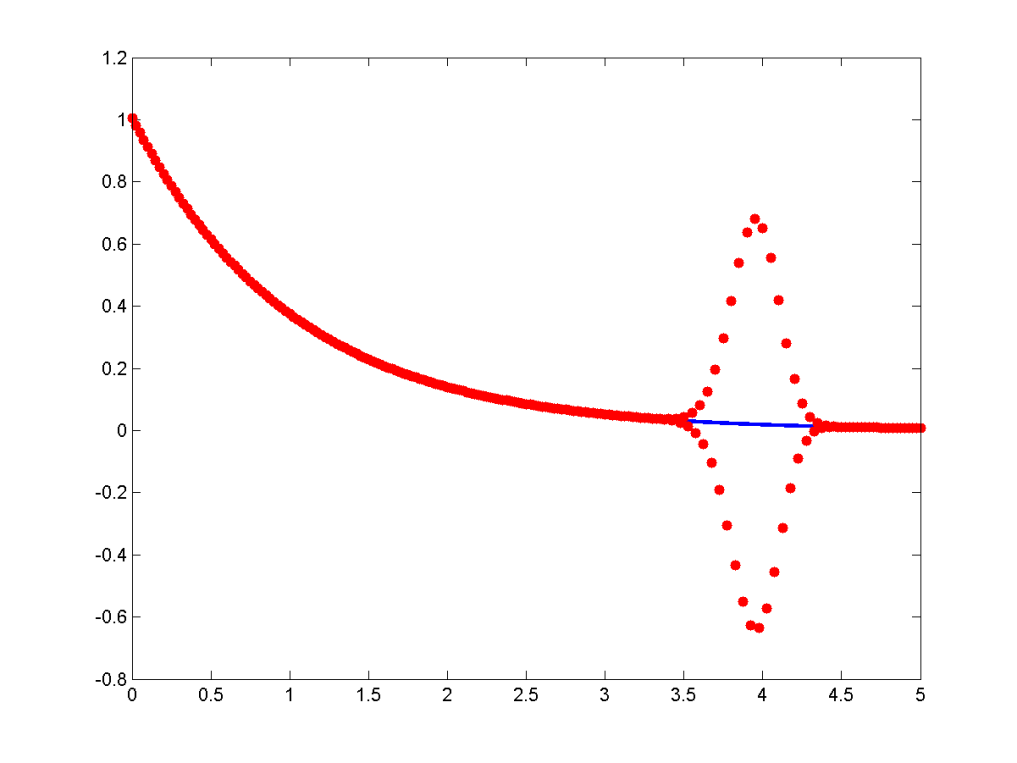
$A = 5,\ T = 20,\ r = 1,\ k = h = 0.05.$



Another

Upwind method

$A = 5,\ T = 40,\ r = 1,\ k = h = 0.025.$



Another

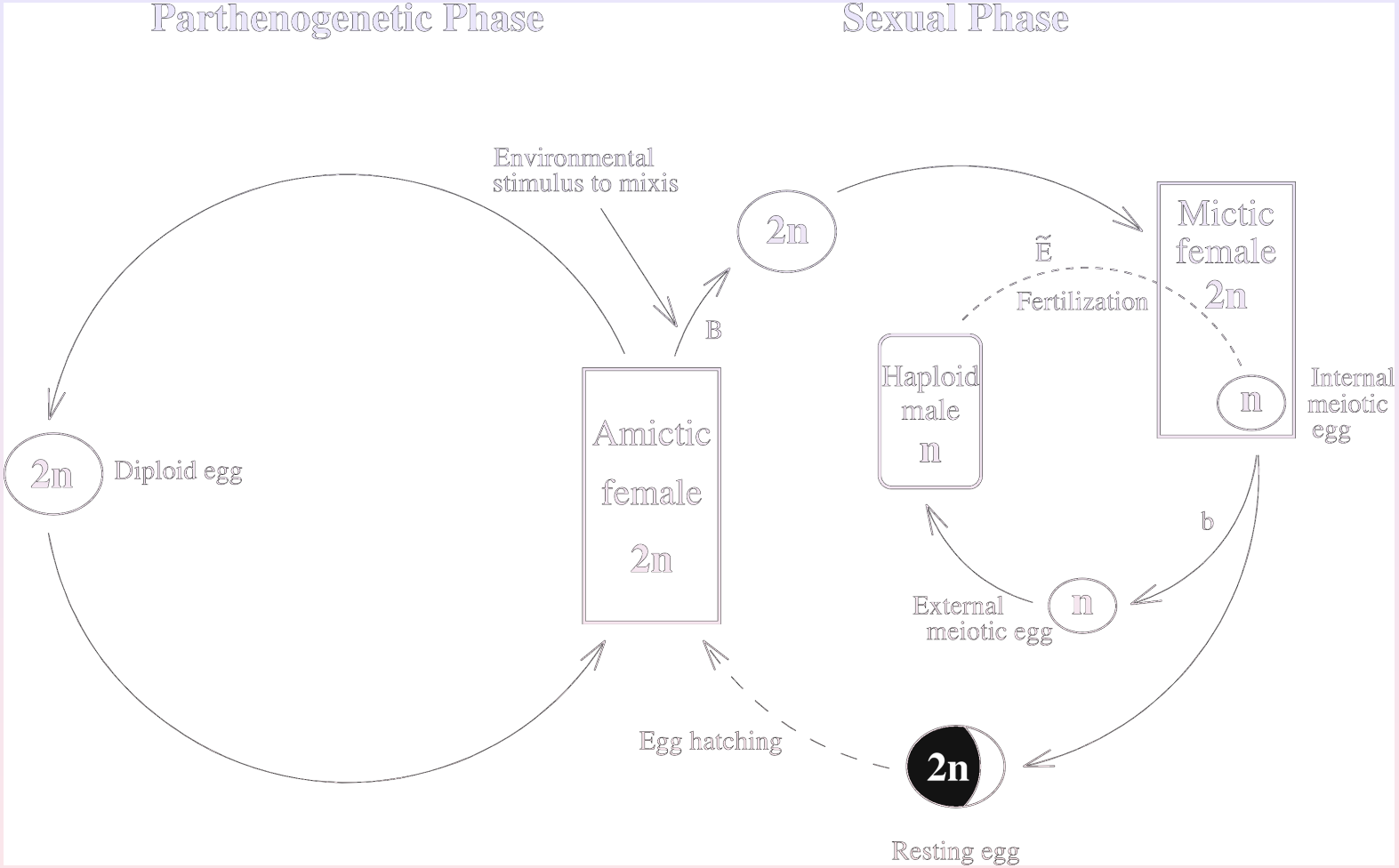
Outline

2. Bounded Mortality — Infinite age
   - Gurtin-MacCamy models
   - Spurious oscilations
   - Reproductive Cycle of Monogonont Rotifera

We are looking for ...

- Treatment of discontinuities.
- The compatibility conditions
- The long time behaviour

Reproductive Cycle of Monogonont Rotifera



Calsina & Ripoll (2002)

Nondimensionalized Model for the Sexual Phase

$$\begin{cases} v_t(a,t) + v_a(a,t) + \mu\, v(a,t) = -E\, H(t)\, v(a,t)\, \chi_{[0,T]}(a) \\ h_t(a,t) + h_a(a,t) + \delta\, h(a,t) = 0 \\ v(0,t) = 1 \\ h(0,t) = \displaystyle\int_1^\infty v(a,t)\, da \\ v(a,0) = v_0(a) \\ h(a,0) = h_0(a) \end{cases}$$

$$\mu, \delta, E > 0, \qquad 0 < T \leq 1, \qquad H(t) = \int_0^\infty h(a,t)\, da$$

$$\alpha = M\, a;\ \ \tau = M\, t;\ \mu = \tilde{\mu}\, M;\ \delta = \tilde{\delta}\, M;\ E = \tilde{E}\, B\, b\, M^3;\ T = \tilde{T}/M$$

$$\tilde{v}(\alpha,\tau) = B\, v(a,t); \quad \tilde{h}(\alpha,\tau) = B\, b\, M\, h(a,t);$$

Calsina & Ripoll (2002)

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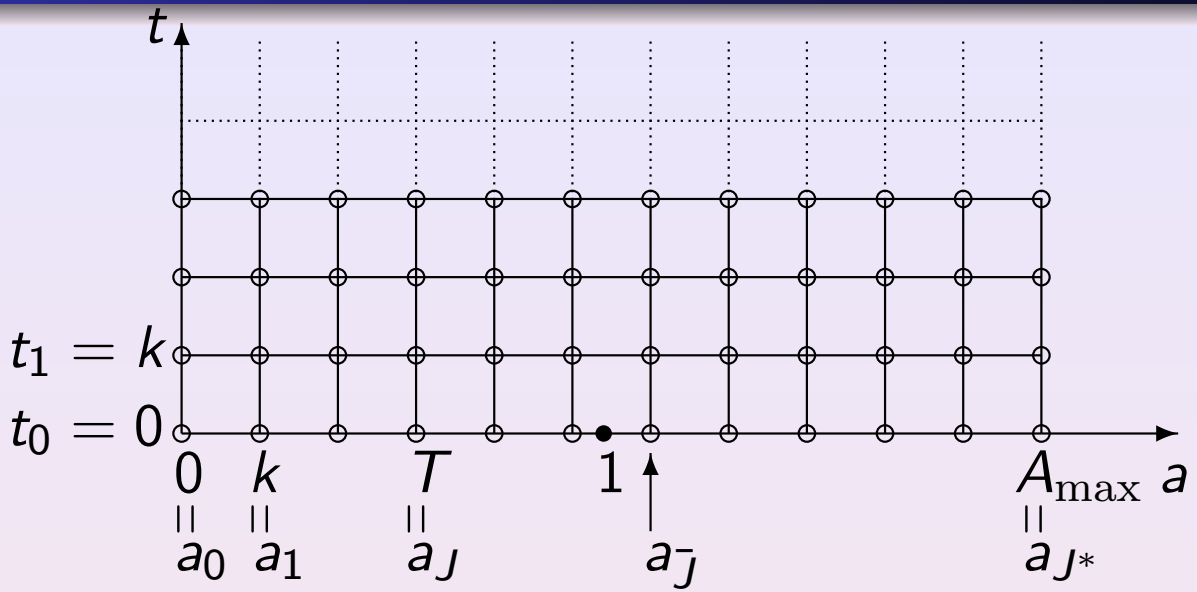
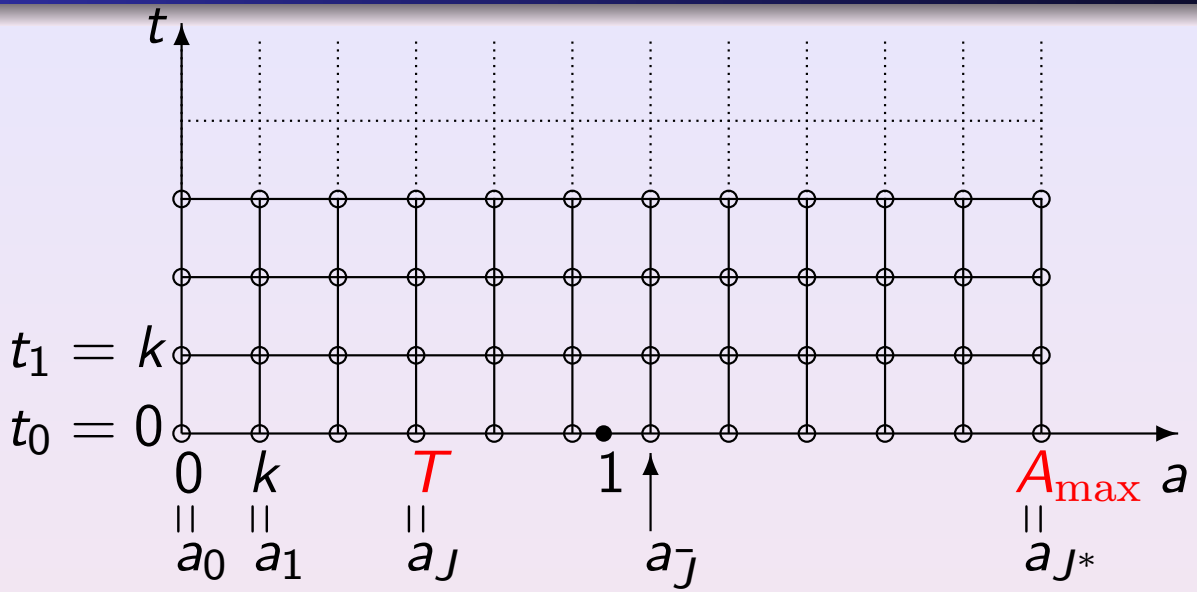
Numerical Method



Parameters

- Introduce $A_{\max}$ (Suppose an integer multiple of T).
- Given $J \in \mathbb{N}$, define the step size $k = T/J$.
- Denote $J^* = A_{\max}/k$.
- Grid points $a_j = j\,k$, $j = 0, 1, \ldots, J^*$ ($a_J = T$, $a_{J^*} = A_{\max}$).
- Denote $\bar{J}$ the first nonnegative integer such that $1 < a_{\bar{J}}$.
- Time levels $t_n = n\,k$, $n = 0, 1, \ldots$

Numerical Method



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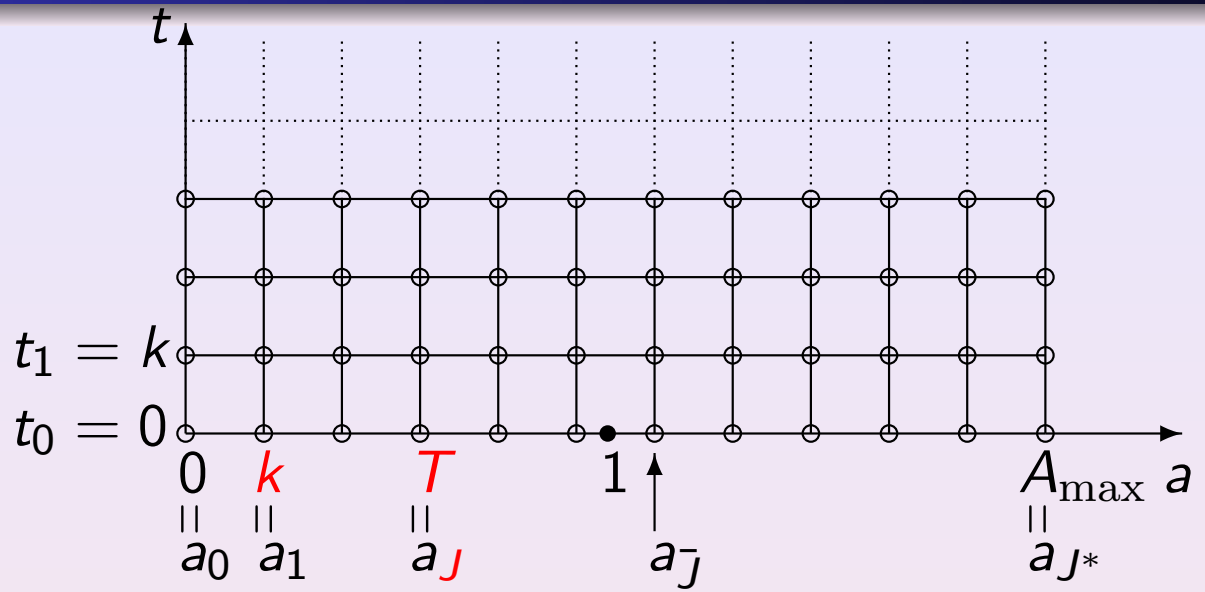

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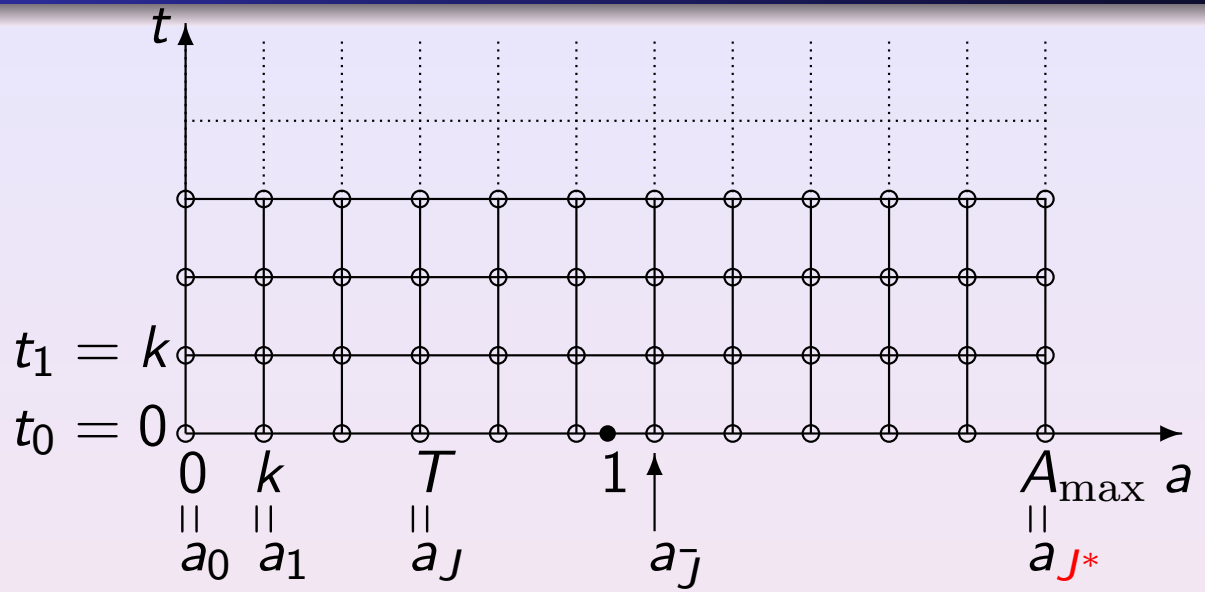
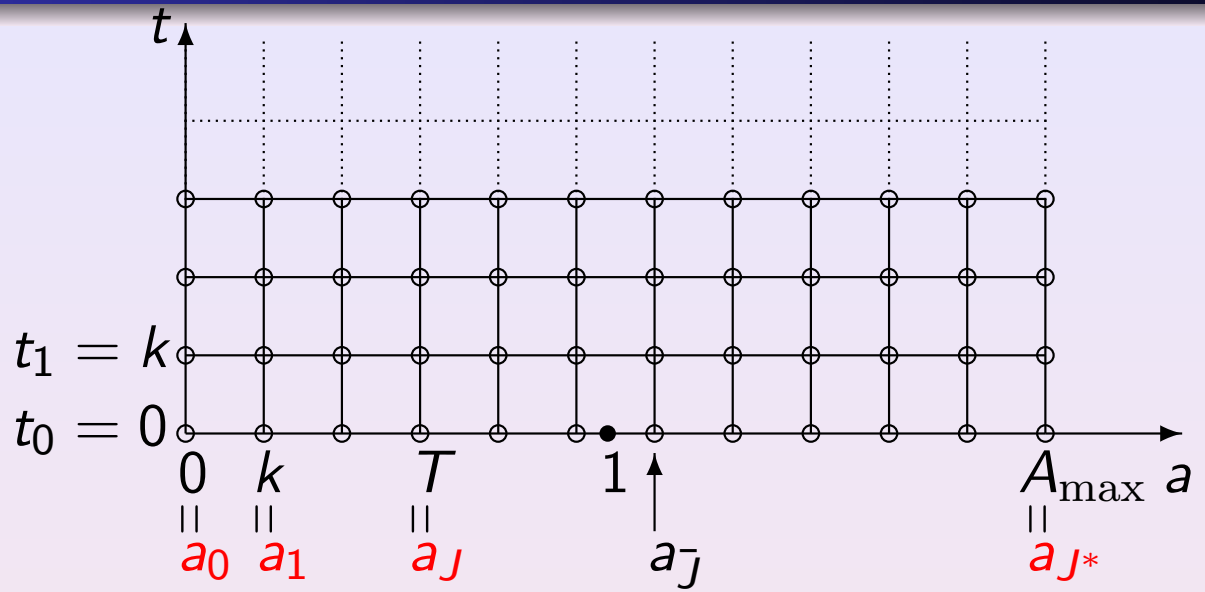


Parameters

- Introduce $A_{\max}$ (Suppose an integer multiple of T).
- Given $J \in \mathbb{N}$, define the step size $k = T/J$.
- Denote $J^* = A_{\max}/k$.
- Grid points $a_j = j\,k$, $j = 0, 1, \ldots, J^*$ ($a_J = T$, $a_{J^*} = A_{\max}$).
- Denote $\bar{J}$ the first nonnegative integer such that $1 < a_{\bar{J}}$.
- Time levels $t_n = n\,k$, $n = 0, 1, \ldots$

Numerical Method



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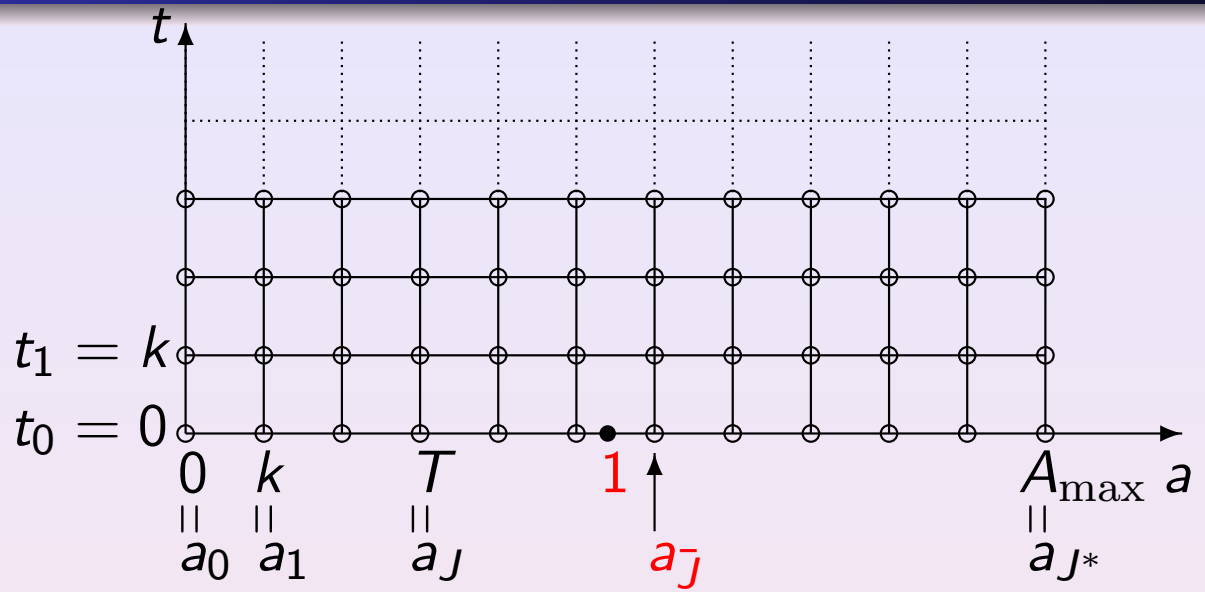
Numerical Method



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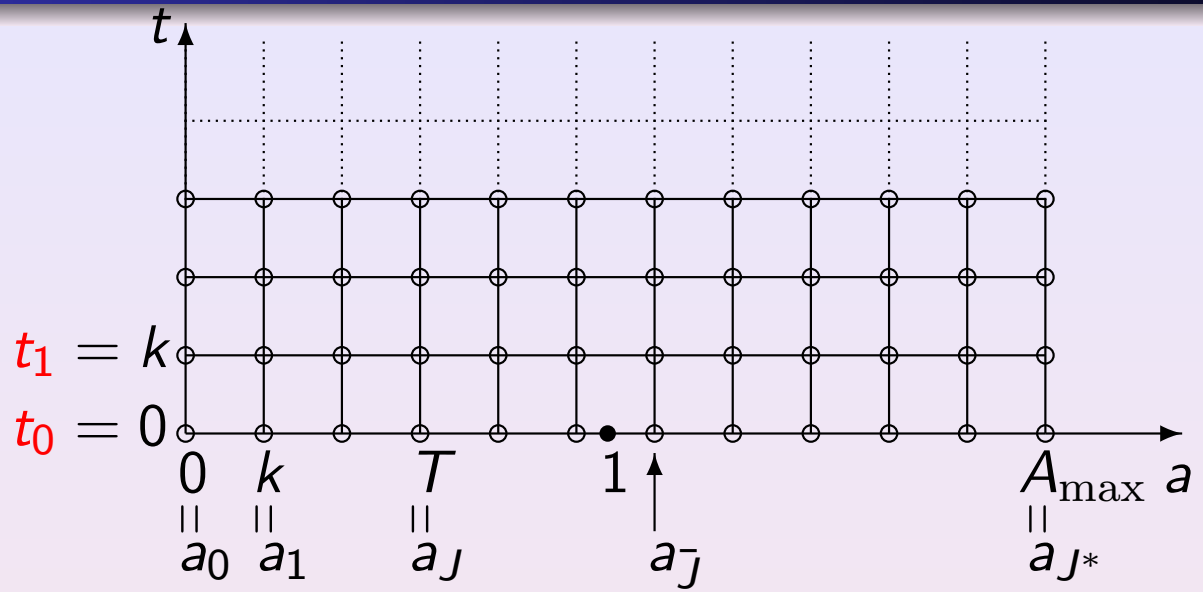
Numerical Method



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- Time levels $t_n = n\,k$, $n = 0, 1, \ldots$

Integration along the characteristic curves

$$\begin{cases} v_t(a,t) + v_a(a,t) + \mu\, v(a,t) &= -E\, H(t)\, v(a,t)\, \chi_{[0,T]}(a) \\ h_t(a,t) + h_a(a,t) + \delta\, h(a,t) &= 0 \end{cases}$$

Integral representation along characteristics

$$\begin{cases} v(a+k,t+k) &= v(a,t)\, \mathrm{e}^{-\int_0^k \left(\mu + E\, H(t+\tau)\, \chi_{[0,T]}(a+\tau)\right) d\tau} \\ h(a+k,t+k) &= h(a,t)\, \mathrm{e}^{-\delta\, k} \end{cases}$$

Grid restriction and discretization

$$\begin{cases} V_{j+1}^{n+1} &= V_j^n\, \mathrm{e}^{-\frac{k}{2}\left(2\,\mu + E\left(I_k(\mathbf{H}^n) + I_k(\mathbf{H}^{n+1})\right)\right)}, & 0 \le j \le J-1 \\ V_{j+1}^{n+1} &= V_j^n\, \mathrm{e}^{-\mu\, k}, & J \le j \le J^*-1 \\ H_{j+1}^{n+1} &= H_j^n\, \mathrm{e}^{-\delta\, k}, & 0 \le j \le J^*-1 \end{cases}$$

$$I_k(\mathbf{H}^n) = \sum_{j=1}^{J^*} \frac{k}{2}\left(H_{j-1}^n + H_j^n\right), \quad I_k(\mathbf{H}^{n+1}) = \sum_{j=1}^{J^*} \frac{k}{2}\left(H_{j-1}^{n+1} + H_j^{n+1}\right)$$

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Discretization of the boundary condition

$$\begin{cases} v(0,t) &= 1 \\ h(0,t) &= \displaystyle\int_1^\infty v(a,t)\,da \end{cases}$$

Grid restriction and discretization

$$\begin{cases} V_0^{n+1} &= 1 \\ H_0^{n+1} &= (a_{\bar{J}} - 1)\, V_{\bar{J}}^{n+1} + \displaystyle\sum_{j=\bar{J}+1}^{J^*} \frac{k}{2}\left(V_{j-1}^{n+1} + V_j^{n+1}\right) \end{cases}$$

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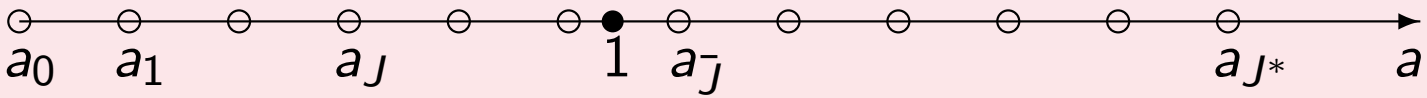

Numerical Method

- Initial Condition: $(\mathbf{V}^0, \mathbf{H}^0)$

$$\begin{aligned} V_j^0 &= v_0(a_j), \quad 0 \le j \le J^* \\ H_j^0 &= h_0(a_j), \quad 0 \le j \le J^* \end{aligned}$$

- General recursion: $(\mathbf{V}^n, \mathbf{H}^n) \to (\mathbf{V}^{n+1}, \mathbf{H}^{n+1})$

$$\begin{aligned} V_0^{n+1} &= 1 \\ V_{j+1}^{n+1} &= V_j^n \, \mathrm{e}^{-\frac{k}{2}\left(2\,\mu + E\left(I_k(\mathbf{H}^n) + I_k(\mathbf{H}^{n+1})\right)\right)}, \; 0 \le j \le J-1 \\ V_{j+1}^{n+1} &= V_j^n \, \mathrm{e}^{-\mu\, k}, \quad J \le j \le J^* - 1 \\ H_0^{n+1} &= (a_{\bar{J}} - 1)\, V_{\bar{J}}^{n+1} + \sum_{j=\bar{J}+1}^{J^*} \frac{k}{2}\left(V_{j-1}^{n+1} + V_j^{n+1}\right) \\ H_{j+1}^{n+1} &= H_j^n \, \mathrm{e}^{-\delta\, k}, \quad 0 \le j \le J^* - 1 \end{aligned}$$

Numerical Method

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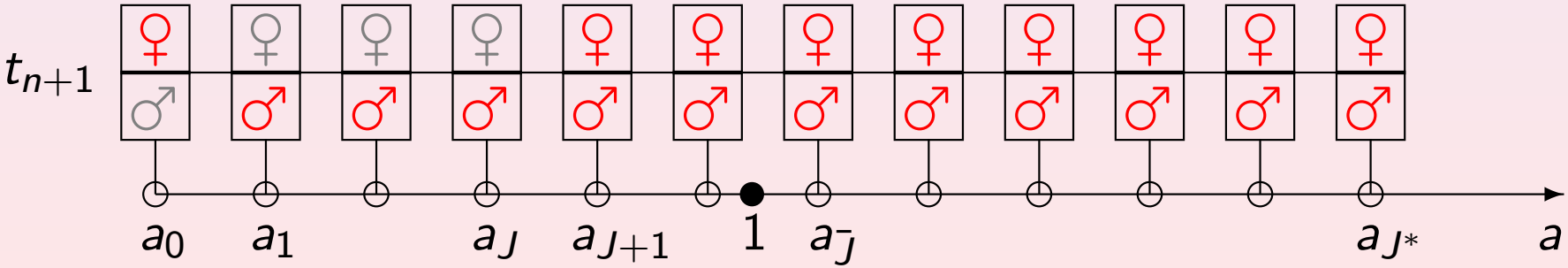

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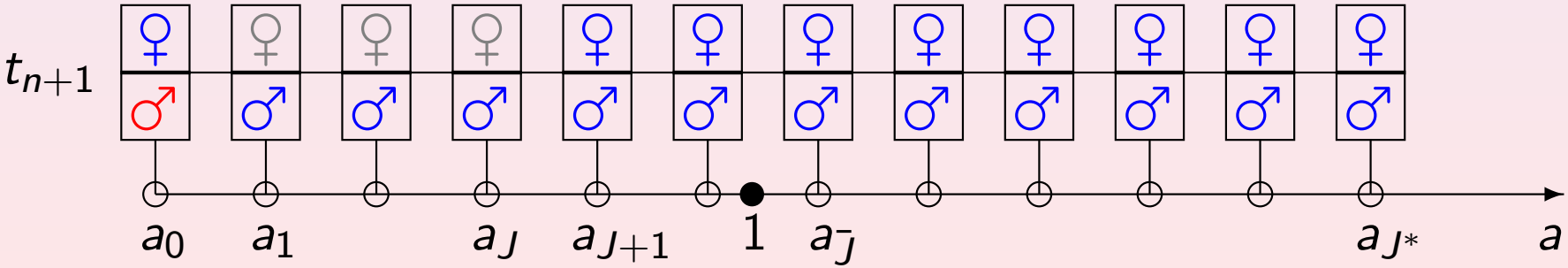

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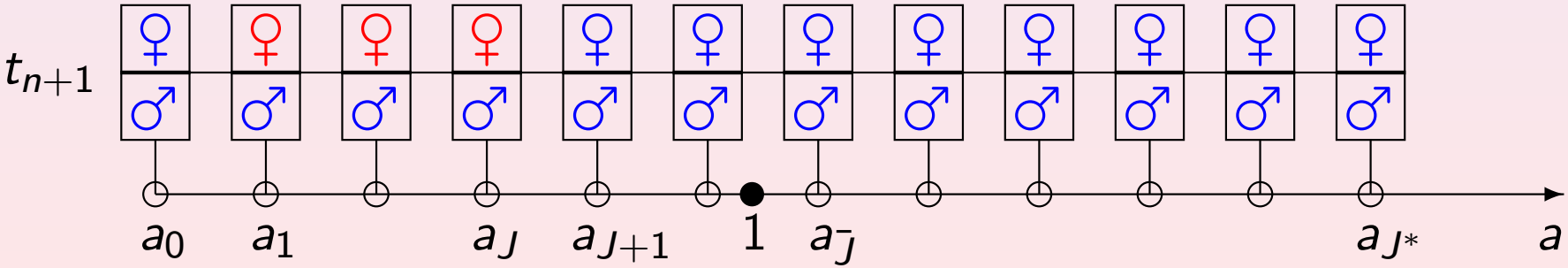

Asymptotic behaviour

Theorem (*Calsina & Ripoll (2002)*)

For any positive values of μ, δ, E and T, there is a unique stationary solution (v^*, h^*) of the model given by

$$\begin{aligned} v^*(a) &= \begin{cases} \mathrm{e}^{-(\mu+E\,H^*)\,a}, & a \in [0,\,T] \\ \mathrm{e}^{-(\mu\,a+E\,H^*\,T)}, & a \in [T,\infty) \end{cases} \\ h^*(a) &= \delta\,H^*\,\mathrm{e}^{-\delta\,a}, \quad a \in [0,\infty) \end{aligned}$$

where H^* (the total population of haploid males at equilibrium) is the unique positive solution of the trascendental equation

$$\mu\,\delta\,H^* = \mathrm{e}^{-(\mu+E\,H^*\,T)}$$

The total population of virgin females at the equilibrium is

$$V^* = \frac{\mu + E\,H^*\,\mathrm{e}^{-(\mu+E\,H^*)\,T}}{\mu\,(\mu + E\,H^*)}$$

Asymptotic behaviour

Theorem (Equilibrium Solution of the Numerical Method)

For any positive values of μ, δ, E, T, $A_{\max}$ and k there is a unique stationary solution $(\mathbf{V}, \mathbf{H})$ of the numerical system given by

$$\begin{aligned} V_j &= \begin{cases} \mathrm{e}^{-(\mu + E\, I_k(\mathbf{H}))\, a_j}, & 0 \le j \le J \\ \mathrm{e}^{-(\mu\, a_j + E\, I_k(\mathbf{H})\, T)}, & J+1 \le j \le J^* \end{cases} \\ H_j &= H_0\, \mathrm{e}^{-\delta\, a_j}, \quad 0 \le j \le J^* \end{aligned}$$

where $I_k(\mathbf{H})$ (the numerical total population of haploid males at the equilibrium) is the unique positive solution of the transcendental equation

$$\begin{aligned} \mu\, \delta\, I_k(\mathbf{H}) &= \mathrm{e}^{-(\mu\, a_{\bar{J}} + E\, I_k(\mathbf{H})\, T)} \left(1 - \mathrm{e}^{-\delta\, A_{\max}}\right) \frac{\delta\, k}{2} \coth\left(\frac{\delta\, k}{2}\right) \\ &\quad \left\{ \mu\, (a_{\bar{J}} - 1) + \left(1 - \mathrm{e}^{-\mu\, (A_{\max} - a_{\bar{J}})}\right) \frac{\mu\, k}{2} \coth\left(\frac{\mu\, k}{2}\right) \right\} \end{aligned}$$

Asymptotic behaviour

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For any positive values of μ, δ, E, T, $A_{\max}$ and k there is a unique stationary solution $(\mathbf{V}, \mathbf{H})$ of the numerical system given by

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H_j &= H_0\, \mathrm{e}^{-\delta\, a_j}, \quad 0 \le j \le J^*
\end{aligned}
$$

The numerical total population of virgin females at the equilibrium can be written as

$$
\begin{aligned}
I_k(\mathbf{V}) = & \frac{1 - \mathrm{e}^{-(\mu + E I_k(\mathbf{H}))\, T}}{\mu + E I_k(\mathbf{H})} \frac{(\mu + E I_k(\mathbf{H}))\, k}{2} \coth\left(\frac{(\mu + E I_k(\mathbf{H}))\, k}{2}\right) \\
& + \frac{1}{\mu}\, \mathrm{e}^{-(\mu + E\, I_k(\mathbf{H}))\, T} \left(1 - \mathrm{e}^{-\mu\,(A_{\max} - T)}\right) \frac{\mu\, k}{2} \coth\left(\frac{\mu\, k}{2}\right)
\end{aligned}
$$

Asymptotic behaviour

Theorem (Convergence)

For any positive values of μ, δ, E, T, $A_{\max}$ and k, denote

Original	*Numerical*	*at Equilibrium*
$v^*(a)$	$\mathbf{V}$	*Density of virgin females*
$h^*(a)$	$\mathbf{H}$	*Density fo haploid males*
V^*	$I_k(\mathbf{V})$	*Total population of virgin females*
H^*	$I_k(\mathbf{H})$	*Total population of haploid males*

There is a constant $\lambda > 0$, such that, as $A_{\max} \to +\infty$ and $k \to 0$

$$
\begin{aligned}
|v^*(a_j) - V_j| &= \mathcal{O}(\mathrm{e}^{-\lambda A_{\max}}) + \mathcal{O}(k^2), \quad 0 \le j \le J^* \\
|h^*(a_j) - H_j| &= \mathcal{O}(\mathrm{e}^{-\lambda A_{\max}}) + \mathcal{O}(k^2), \quad 0 \le j \le J^* \\
|V^* - I_k(\mathbf{V})| &= \mathcal{O}(\mathrm{e}^{-\lambda A_{\max}}) + \mathcal{O}(k^2) \\
|H^* - I_k(\mathbf{H})| &= \mathcal{O}(\mathrm{e}^{-\lambda A_{\max}}) + \mathcal{O}(k^2)
\end{aligned}
$$

Numerical Validation

Calsina & Ripoll (2002)

The unique stationary population density is stable as along as the male-female encounter rate remains below a critical value E_{un}. Beyond this critical value the stationary population becomes unstable and a stable limit cycle (isolated periodic orbit) appears

Numerical simulation.

- Stable case.
- Unstable case.

Stable case.

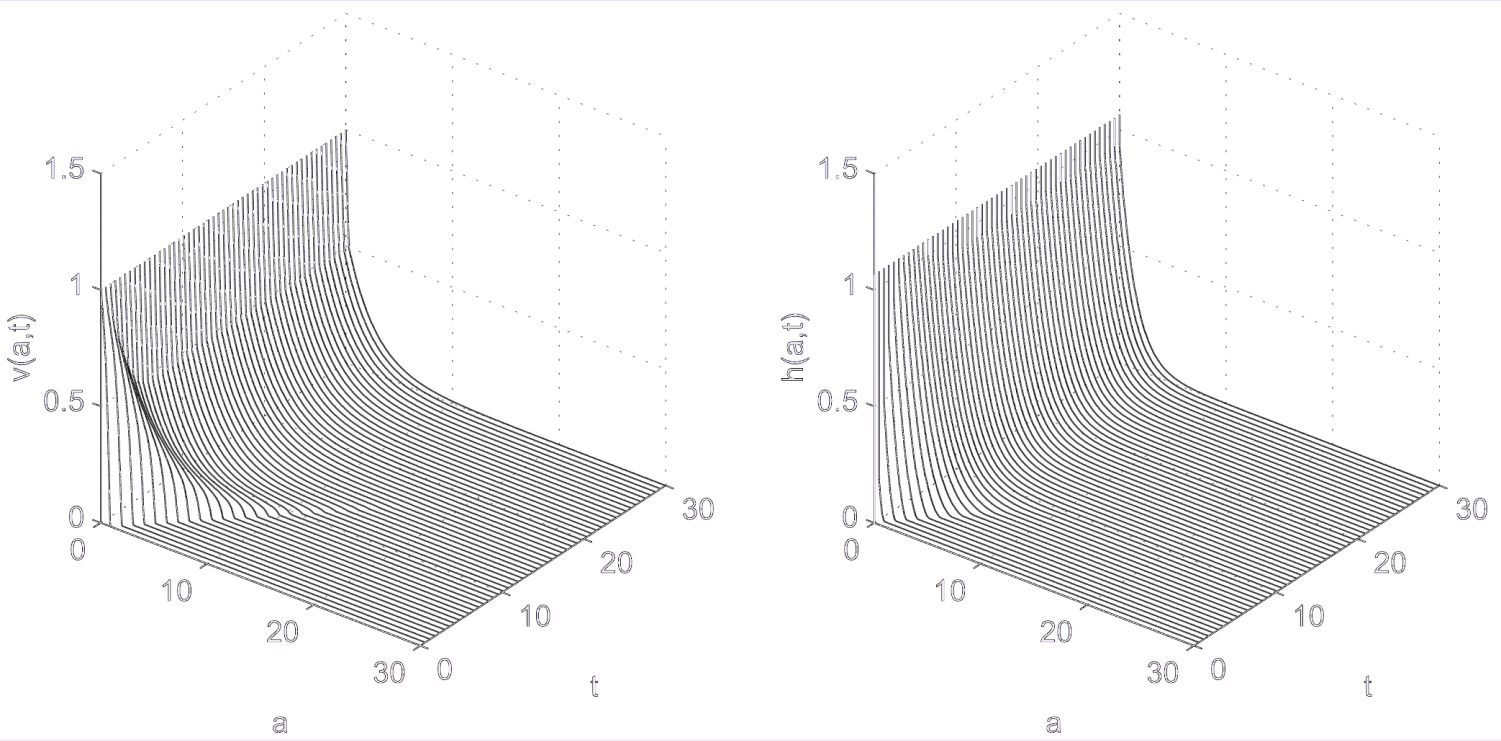
Parameters

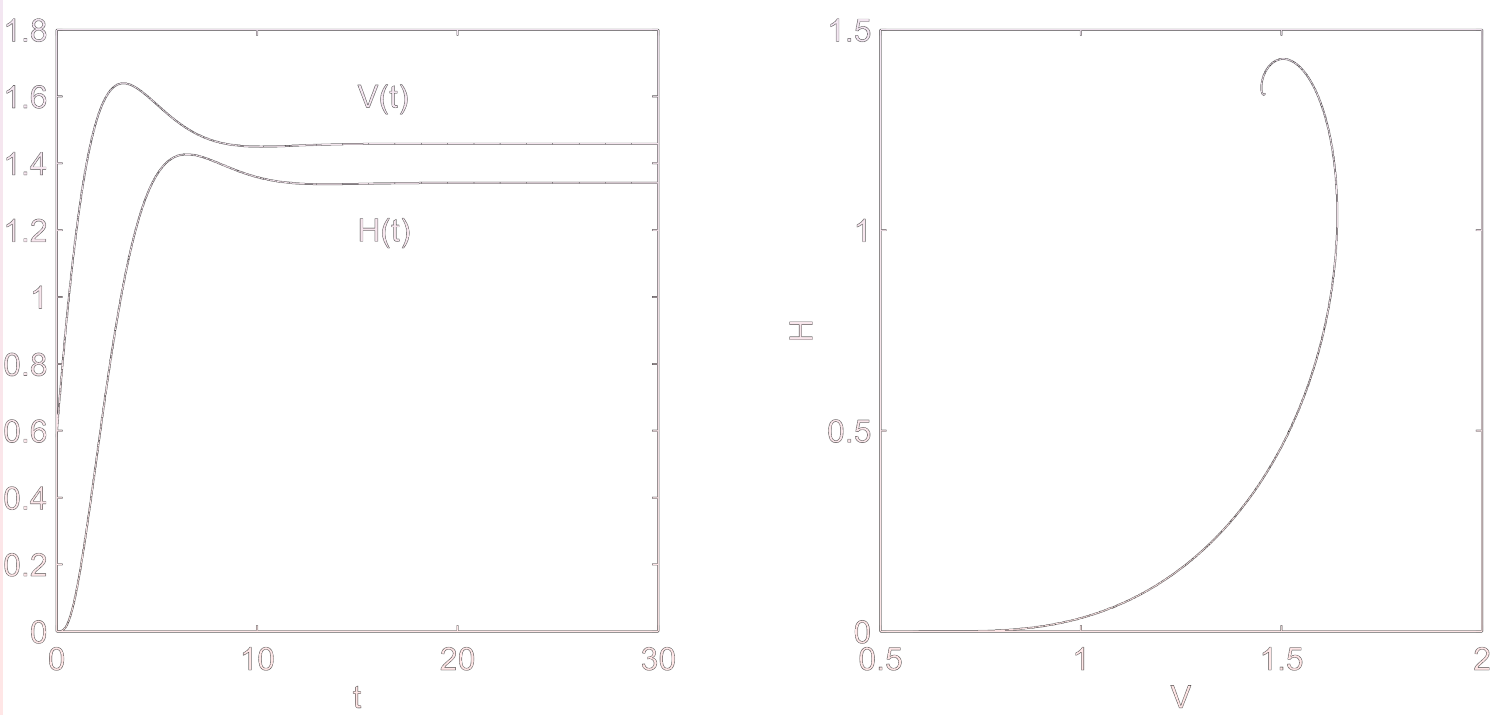
- $\mu = 0.4, \delta = 0.7, E = 1.44, T = 0.3$ ($E_{\mathrm{un}} = 1617.928392$)
- Equilibrium populations: $H^* = 1.341202, \quad V^* = 1.458016$
- $A_{\max} = 39, \quad J = 100 \ \ (k = 3 \times 10^{-3})$

Initial Condition

$$v_0(a) = \begin{cases} \mathrm{e}^{-\mu \frac{a}{1-a}}, & a \in [0,1] \\ 0, & a \in [1, A_{\max}] \end{cases}$$

$$h_0(a) = 0, \quad a \in [0, A_{\max}]$$

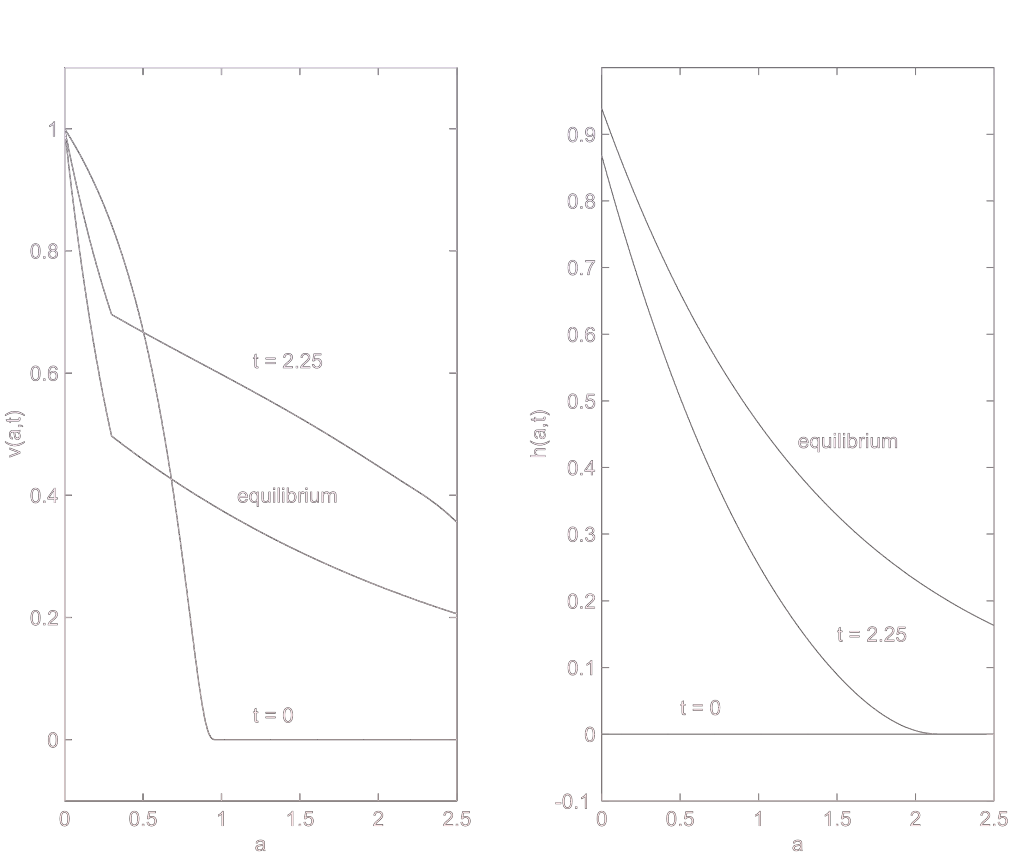

v(a,t)
1.5
1
0.5
0
0
10
20
30
a
0
10
20
30
t
h(a,t)
1.5
1
0.5
0
0
10
20
30
a
0
10
20
30
t


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
V(t)
H(t)
0
10
20
30
t
1.5
1
0.5
0
H
0.5
1
1.5
2
V

Discontinuities.

Compatibility conditions

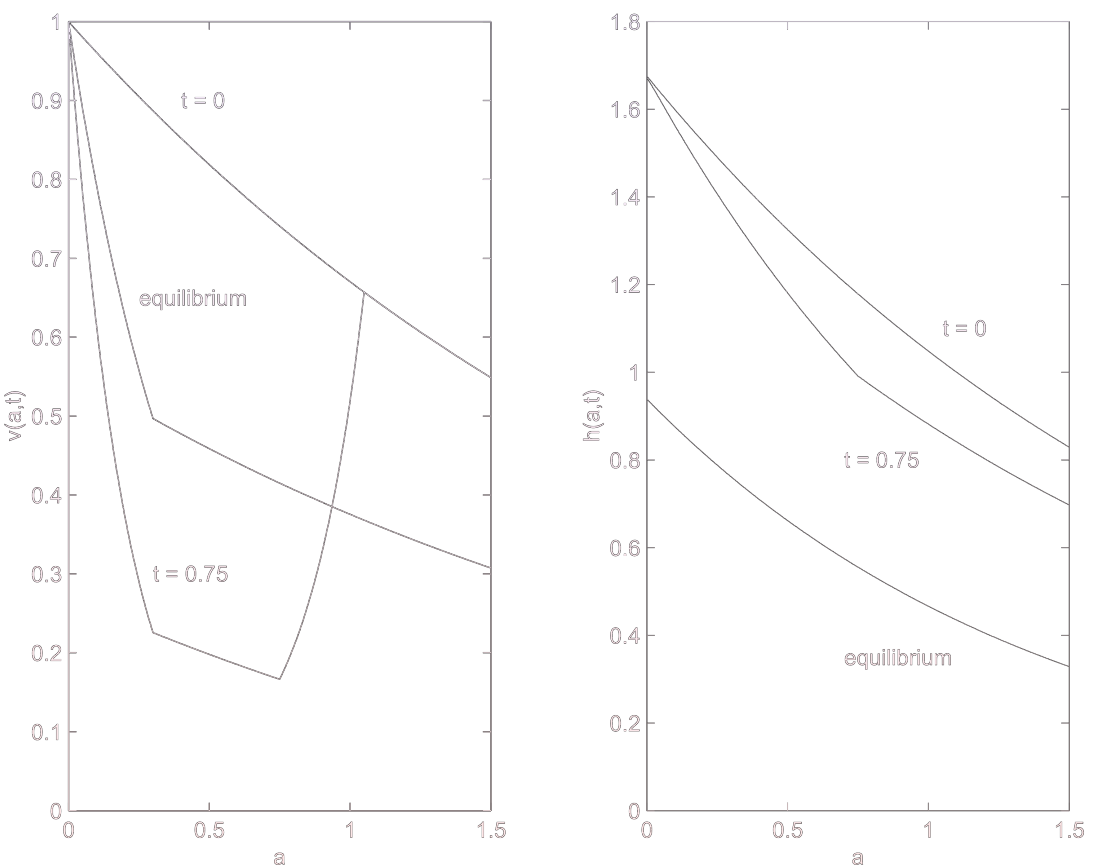
$$v_0'(0) = -(\mu + E\,H(0))\,v_0(0)$$
$$h_0'(0) = -v_0(1) + (\mu - \delta)\,h_0(0)$$



Discontinuities.

Initial condition

$$v_0(a) = \mathrm{e}^{-\mu\, a}, \quad a \in [0, A_{\max}]$$
$$h_0(a) = \frac{1}{\mu}\mathrm{e}^{-\left(\delta\, \mathrm{e}^{-\mu}\, a + \mu\right)}, \quad a \in [0, A_{\max}]$$



Unstable case.

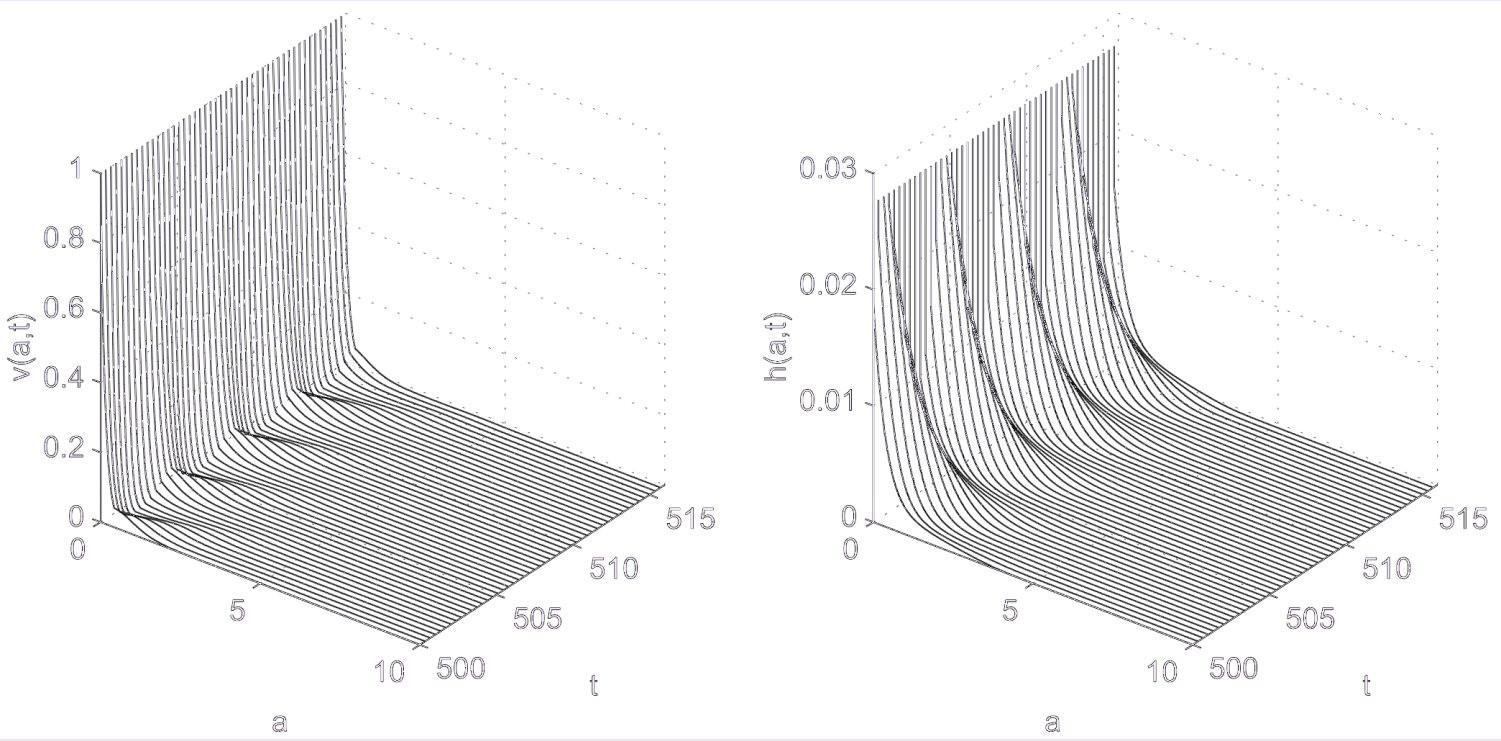
Parameters

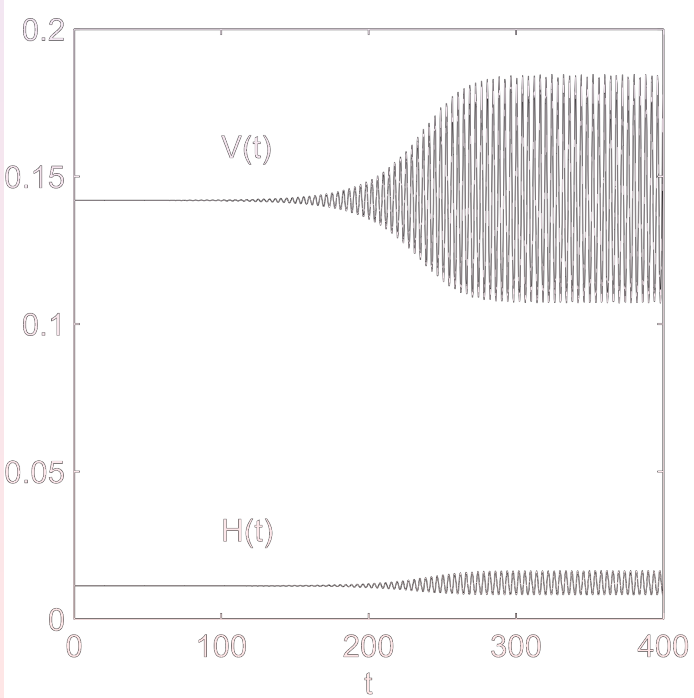
- $\mu = 0.9355, \delta = 1.4463, E = 675.84, T = 0.4274$ $(E_{\mathrm{un}} = 501.831883)$
- Equilibrium populations: $H^* = 0.011259,\ V^* = 0.141869$
- $A_{\max} = 21.37, \quad J = 100\ \ (k = 4.274 \times 10^{-3})$

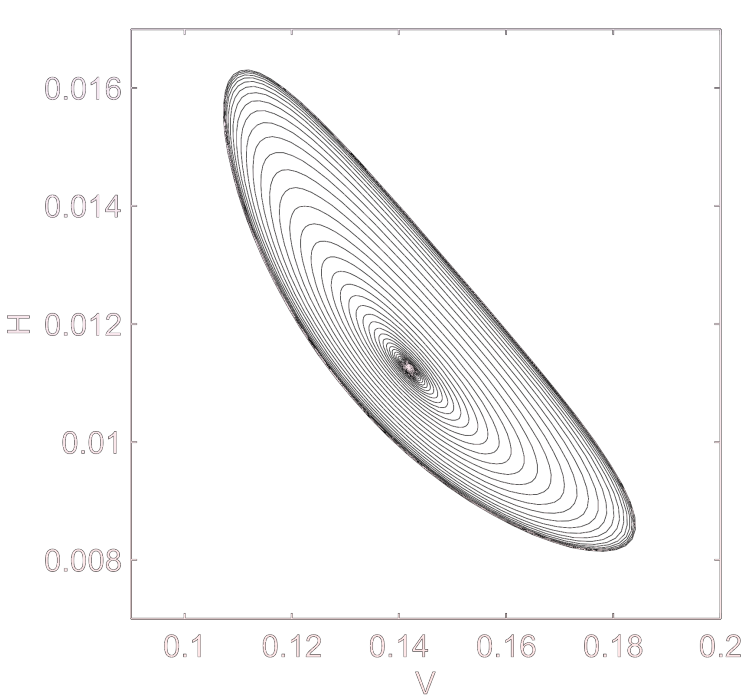
Initial condition

$$v_0(a) = \begin{cases} \mathrm{e}^{-(\mu + E\,H^*)\,a}, & a \in [0, T] \\ \mathrm{e}^{-(\mu\,a + E\,H^*\,T)}, & a \in [T, A_{\max}] \end{cases}$$

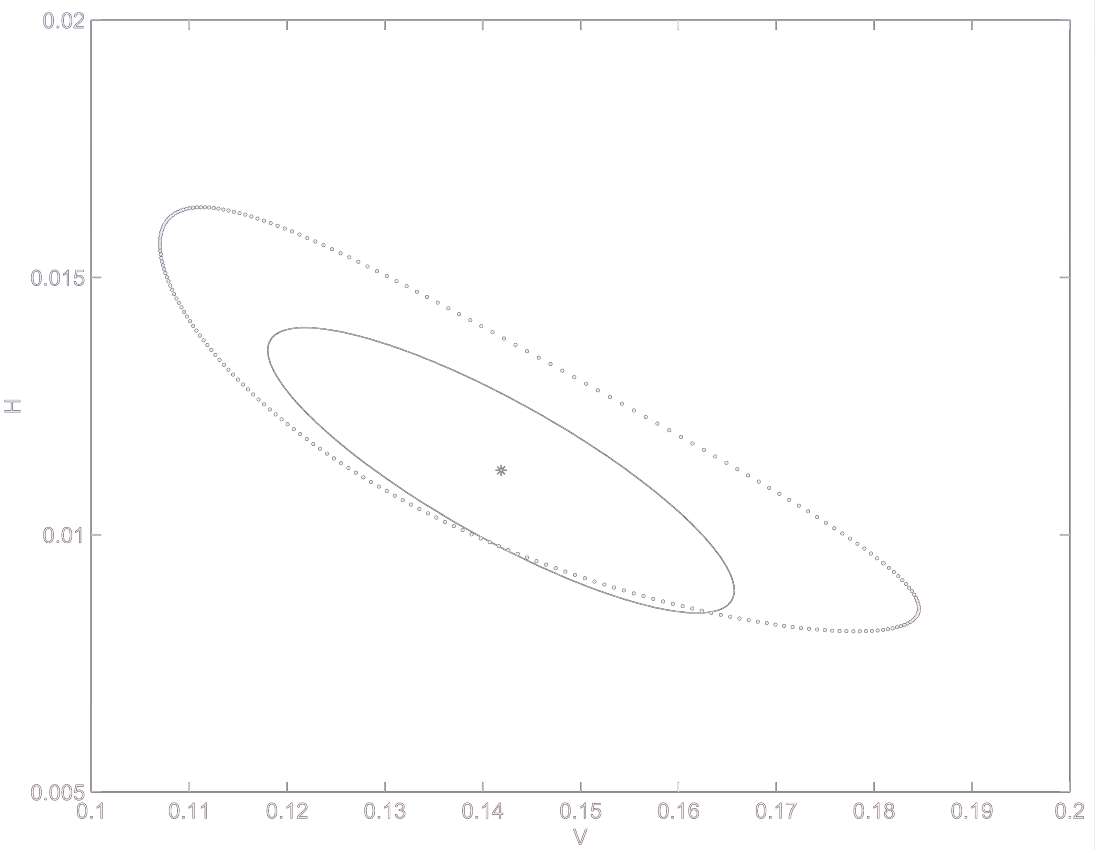
$$h_0(a) = \delta\,H^*\,\mathrm{e}^{-\delta\,a}, \quad a \in [0, A_{\max}]$$


v(a,t)
1
0.8
0.6
0.4
0.2
0
0
5
10
a
500
505
510
515
t
h(a,t)
0.03
0.02
0.01
0
0
5
10
a
500
505
510
515
t


0.2
0.15
0.1
0.05
0
V(t)
H(t)
0
100
200
300
400
t


0.016
0.014
0.012
0.01
0.008
H
0.1
0.12
0.14
0.16
0.18
0.2
V

Limit cicle



(*) Equilibrium

(-) Limit cicle obtained by Calsina & Ripoll (2002) (period 3.9163)

(.) Limit cicle obtained by the numerical method (period 3.9857)

Part III

Size-Structured Population Models

Outline

3. Non Hierarchical models
   - Models and Numerical analysis
   - Dynamics of Gambussia affinis Population

Size-Structured population model

$$u_t + (g(x, I_g(t), t)\, u)_x = -\mu(x, I_\mu(t), t)\, u, \quad x_{\min} < x < x_{\max}, \quad t > 0,$$

$$g(x_{\min}, I_g(t), t)\, u(x_{\min}, t) = \int_{x_{\min}}^{x_{\max}} \alpha(x, I_\alpha(t), t)\, u(x, t)\, dx, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad x_{\min} \le x \le x_{\max},$$

$$I_\phi(t) = \int_{x_{\min}}^{x_{\max}} \gamma_\phi(x)\, u(x, t)\, dx, \quad t \ge 0, \quad \phi = \alpha, \mu, g.$$

L. M. Abia, O. Angulo and J. C. López-Marcos,
Size-structured population dynamics models and their numerical solutions,
Discrete Contin. Dyn. Syst. B, **4** (2004) 1203-1222.

Finite Difference Methods

- $J \in \mathbb{N}$, $x_j = x_{\min} + j\,h$, $0 \le j \le J$, $h = \dfrac{x_{\max} - x_{\min}}{J}$,
  $t_n = n\,k$, $0 \le n \le N$, $N = \left[\dfrac{T}{k}\right]$.
- Discretization of

$$u_t + (g(x, I_g(t), t)\,u)_x = -\mu(x, I_\mu(t), t)\,u.$$

- Boundary condition discretization

$$g(x_{\min}, I_g(t), t)\,u(x_{\min}, t) = \int_{x_{\min}}^{x_{\max}} \alpha(x, I_\alpha(t), t)\,u(x, t)\,dx.$$

- Initial condition on the grid.

References

D. Sulsky,
Numerical solution of structured population models II. Mass structure,
J. Math. Biol., **32** (1994) 491–514.
Lax-Wendroff method.

O. Angulo and J.C. López-Marcos,
Numerical integration of autonomous and nonautonomous nonlinear size-structured population models,
Math. Biosci., **177-178** (2002) 39–71.
Lax-Wendroff and box methods.

O. Angulo,
Estudio Numérico de Modelos de Poblaciones Estructuradas por el Tamaño, PhD thesis, Universidad de Valladolid, 2002.
Lax-Wendroff and box methods.

Integration along characteristics. Theoretical representation of the solution

$$\mu^*(x, z_1, z_2, t) = \mu(x, z_1, t) + g_x(x, z_2, t),$$

$$u_t(x,t) + g(x, l_g(t), t)\, u_x(x,t) = -\mu^*(x, l_\mu(t), l_g(t), t)\; u(x,t), \quad \mathrm{x_{min}} < x < \mathrm{x_{max}}.$$

- $x(t; t_0, x_0)$,

$$\begin{cases} x'(t; t_0, x_0) = g(x(t; t_0, x_0), l_g(t), t), & t \geq t_0, \\ x(t_0; t_0, x_0) = x_0. \end{cases}$$

- $w(t; t_0, x_0) = u(x(t; t_0, x_0), t),\ t \geq t_0.$

Integration along characteristics. Theoretical representation of the solution

$$u_t(x,t)+g(x,I_g(t),t)\,u_x(x,t) = -\mu^*(x,I_\mu(t),I_g(t),t)\;u(x,t), \quad \mathrm{x_{min}} < x < \mathrm{x_{max}}.$$

- $x(t;t_0,x_0)$,

$$\begin{cases} x'(t;t_0,x_0) = g(x(t;t_0,x_0),I_g(t),t), & t \geq t_0, \\ x(t_0;t_0,x_0) = x_0. \end{cases}$$

- $w(t;t_0,x_0) = u(x(t;t_0,x_0),t)$, $t \geq t_0$.

$$\begin{cases} \dfrac{d}{dt}w(t;t_0,x_0) &= -\mu^*\left(x\left(t;t_0,x_0\right),I_\mu(t),I_g(t),t\right)\;w(t;t_0,x_0), \quad t \geq t_0, \\ w(t_0;t_0,x_0) &= u(x_0,t_0), \end{cases}$$

$$w(t;t_0,x_0) = u(x_0,t_0)\,\exp\left\{-\int_{t_0}^{t}\mu^*\left(x\left(\tau;t_0,x_0\right),I_\mu(\tau),I_g(\tau),\tau\right)\,d\tau\right\}, \quad t \geq t_0.$$

Integration along characteristics. Theoretical representation of the solution

$$u_t(x,t)+g(x,I_g(t),t)\,u_x(x,t) = -\mu^*(x,I_\mu(t),I_g(t),t)\;u(x,t), \quad \mathrm{x_{min}} < x < \mathrm{x_{max}}.$$

- $x(t;t_0,x_0)$,

$$\begin{cases} x'(t;t_0,x_0) = g(x(t;t_0,x_0),I_g(t),t), & t \geq t_0, \\ x(t_0;t_0,x_0) = x_0. \end{cases}$$

- $w(t;t_0,x_0) = u(x(t;t_0,x_0),t)$, $t \geq t_0$.

$$w(t;t_0,x_0) = u(x_0,t_0)\,\exp\left\{-\int_{t_0}^{t} \mu^*\left(x\left(\tau;t_0,x_0\right),I_\mu(\tau),I_g(\tau),\tau\right)\,d\tau\right\}, \quad t \geq t_0.$$

Characteristics methods

- $J \in \mathbb{N}$, $X_j^0 = \mathrm{x}_{\min} + j\,h$, $0 \le j \le J$, $h = \dfrac{x_{\max} - x_{\min}}{J}$,
  $t_n = n\,k$, $0 \le n \le N$, $N = \left[\dfrac{T}{k}\right]$.
- Discretization of

$$\begin{cases} x'(t; t_n, X_j^n) = g(x(t; t_n, X_j^n), I_g(t), t), & t \ge t_n, \\ x(t_n; t_n, X_j^n) = X_j^n. \end{cases}$$

$$w(t; t_n, X_j^n) = u(X_j^n, t_n) \exp\left\{ - \int_{t_n}^{t} \mu^* \left(x\left(\tau; t_n, X_j^n\right), I_\mu(\tau), I_g(\tau), \tau \right) \, d\tau \right\}.$$

- Boundary condition discretization

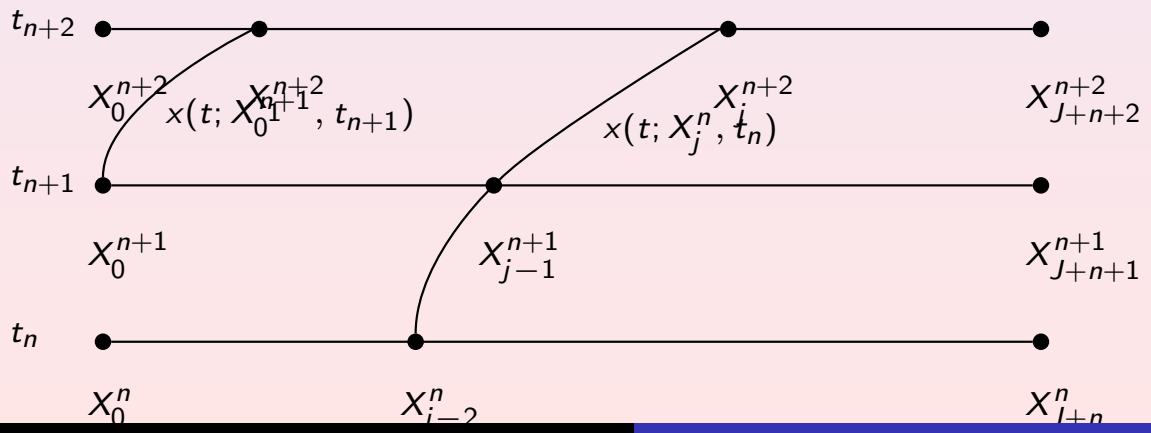
$$g(\mathrm{x}_{\min}, I_g(t), t)\, u(\mathrm{x}_{\min}, t) = \int_{\mathrm{x}_{\min}}^{\mathrm{x}_{\max}} \alpha(x, I_\alpha(t), t)\, u(x, t)\, dx,$$

- Initial condition on the grid.

Agregation Grid Nodes method (AGN)

$$\mathbf{X}^{n+2} = \left\{ X_0^{n+2} = \mathrm{x}_{\min}, X_1^{n+2}, \ldots, X_{J+n+1}^{n+2}, X_{J+n+2}^{n+2} = \mathrm{x}_{\max} \right\}, .$$

$$\begin{aligned} X_1^{n+2} &= X_0^{n+1} + k\, g\left(X_0^{n+1} + \frac{k}{2}\, g(X_0^{n+1}, Q^{n+1}(\mathbf{X}^{n+1}, \gamma_g^{n+1}\, \mathbf{U}^{n+1}), t_{n+1}), \right. \\ &\qquad \frac{3\, Q^{n+1}(\mathbf{X}^{n+1}, \gamma_g^{n+1}\, \mathbf{U}^{n+1}) - Q^{n}(\mathbf{X}^{n}, \gamma_g^{n}\, \mathbf{U}^{n})}{2}, \\ &\qquad \left. t_{n+1} + \frac{k}{2} \right), \\ X_j^{n+2} &= X_{j-2}^{n} + 2\, k\, g(X_{j-1}^{n+1}, Q^{n+1}(\mathbf{X}^{n+1}, \gamma_g^{n+1}\, \mathbf{U}^{n+1}), t_{n+1}), \\ &\qquad 2 \le j \le J + n + 1. \end{aligned}$$



Agregation Grid Nodes method (AGN)

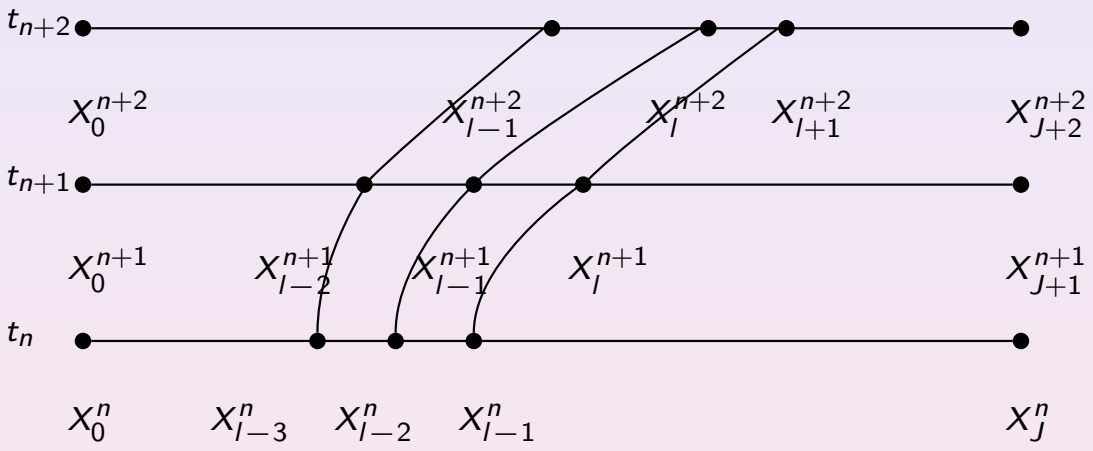
$$\mathbf{X}^{n+2} = \left\{ X_0^{n+2} = \mathrm{x}_{\min}, X_1^{n+2}, \ldots, X_{J+n+1}^{n+2}, X_{J+n+2}^{n+2} = \mathrm{x}_{\max} \right\},$$

$$\mathbf{U}^{n+2} = \left\{ U_0^{n+2}, U_1^{n+2}, \ldots, U_{J+n+1}^{n+2}, U_{J+n+2}^{n+2} = 0 \right\}.$$

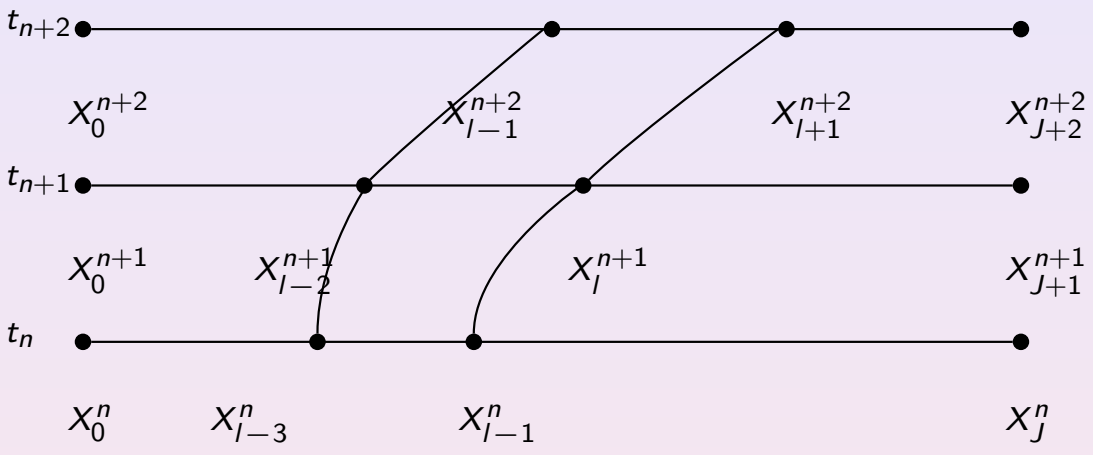
$$\begin{aligned} U_1^{n+2} &= U_0^{n+1} \exp\Bigg(-k\,\mu^* \Bigg(X_0^{n+1} + \frac{k}{2}\, g\left(X_0^{n+1}, Q^{n+1}(\mathbf{X}^{n+1}, \gamma_g^{n+1}\,\mathbf{U}^{n+1}), t_{n+1} \right), \\ &\qquad \frac{3\,Q^{n+1}(\mathbf{X}^{n+1}, \gamma_\mu^{n+1}\,\mathbf{U}^{n+1}) - Q^n(\mathbf{X}^n, \gamma_\mu^n\,\mathbf{U}^n)}{2}, \\ &\qquad \frac{3\,Q^{n+1}(\mathbf{X}^{n+1}, \gamma_g^{n+1}\,\mathbf{U}^{n+1}) - Q^n(\mathbf{X}^n, \gamma_g^n\,\mathbf{U}^n)}{2}, t_{n+1} + \frac{k}{2} \Bigg) \Bigg), \\ U_j^{n+2} &= U_{j-2}^n \exp\Bigg(-2\,k\,\mu^* \Bigg(X_{j-1}^{n+1}, Q^{n+1}\left(\mathbf{X}^{n+1}, \gamma_\mu^{n+1}\,\mathbf{U}^{n+1} \right), \\ &\qquad Q^{n+1}\left(\mathbf{X}^{n+1}, \gamma_g^{n+1}\,\mathbf{U}^{n+1} \right), t_{n+1} \Bigg) \Bigg), \quad 2 \le j \le J+n+1. \end{aligned}$$

$$U_0^n = \frac{Q^n(\mathbf{X}^n, \alpha(\mathbf{X}^n, \mathbf{U}^n)\,\mathbf{U}^n)}{g(X_0^n, Q^n(\mathbf{X}^n, \gamma_g^n\,\mathbf{U}^n), t_n)}.$$

Selection Grid Nodes method (SGN)
Selection Strategy

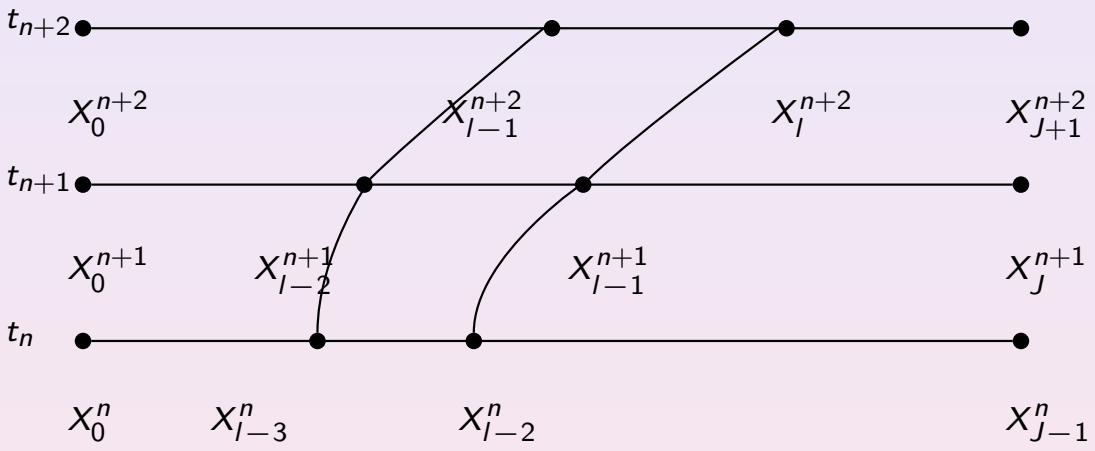


Selection Grid Nodes method (SGN)
Selection Strategy



$$|X_{l+1}^{n+1} - X_{l-1}^{n+1}| = \min_{1 \le j \le J-1} |X_{j+1}^{n+1} - X_{j-1}^{n+1}|.$$

Selection Grid Nodes method (SGN)
Selection Strategy



Convergence

O. Angulo and J. C. López-Marcos,
Numerical integration of fully nonlinear size-structured population models,
Apl. Numer. Maths., **50** (2004) 291-327.

Properties

- CFL condition (Upwind method, Lax-Wendroff method,...)
- Positivity of solutions
- Well defined numerical method
- Consistency
- Nonlinear stability
- Convergence

Outline

3. Non Hierarchical models
   - Models and Numerical analysis
   - Dynamics of Gambussia affinis Population

We are looking for ...

- Comparison of numerical shemes.
- Long time behaviour.

O. Angulo, A. Durán and J. C. López-Marcos,
Numerical study of size-structured population models: A case of Gambusia affinis,
C. R. Biologies, **328** (2005) 387-402.

Gambussia affinis

Preliminaries

- Structured variable: length (mm.), $x \in [9, 63]$.
- The time unit is a day.
- Male and females are equally distributed.
- Competition for food is only taken into account in the mortality function.

Model

$$u_t + (g(x,t)\,u)_x = -\mu(x, I_\mu(t), t)\,u, \quad 9 < x < 63, \quad t > 0,$$

$$g(9,t)\,u(9,t) = \int_9^{63} \alpha(x,t)\,u(x,t)\,dx, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad 9 \le x \le 63,$$

$$I_\mu(t) = \int_9^{63} \gamma_\mu(x)\,u(x,t)\,dx, \quad t \ge 0.$$

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Ecol. Mongr., **18** (1948) 1–43.

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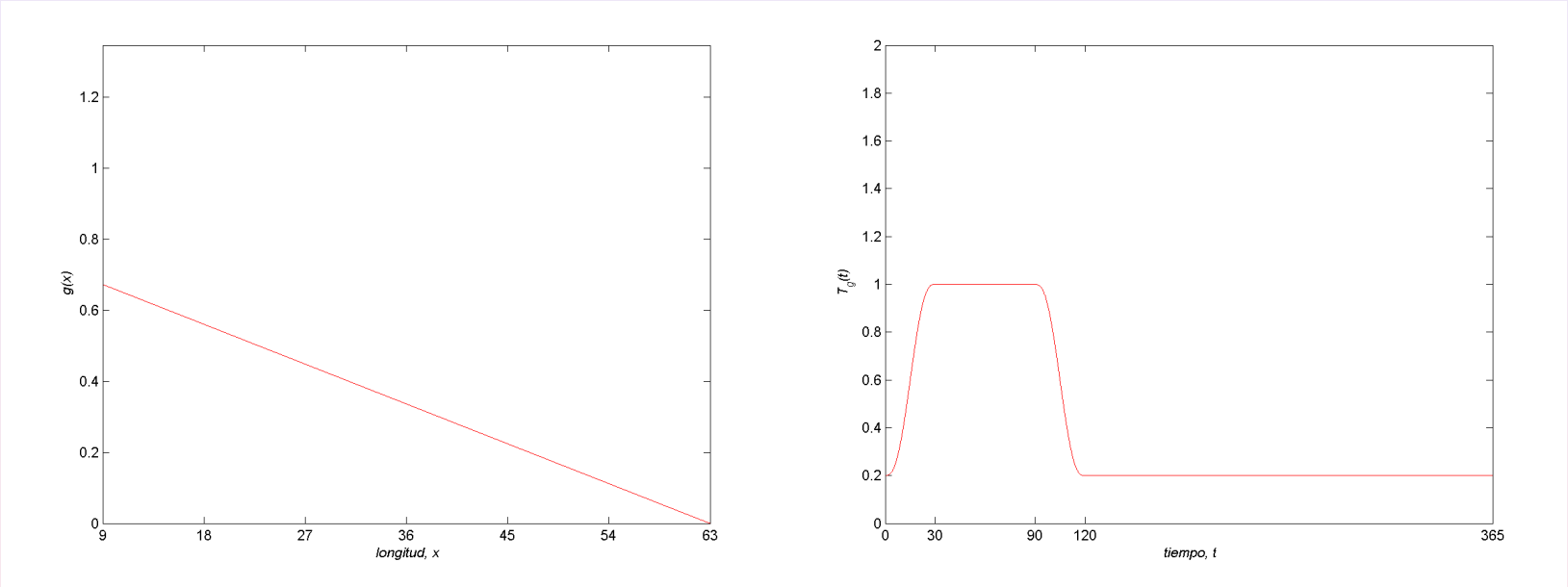
Properties and fitting of the von Bertalanffy growth curve.
Growth, **29** (1965) 265–289.

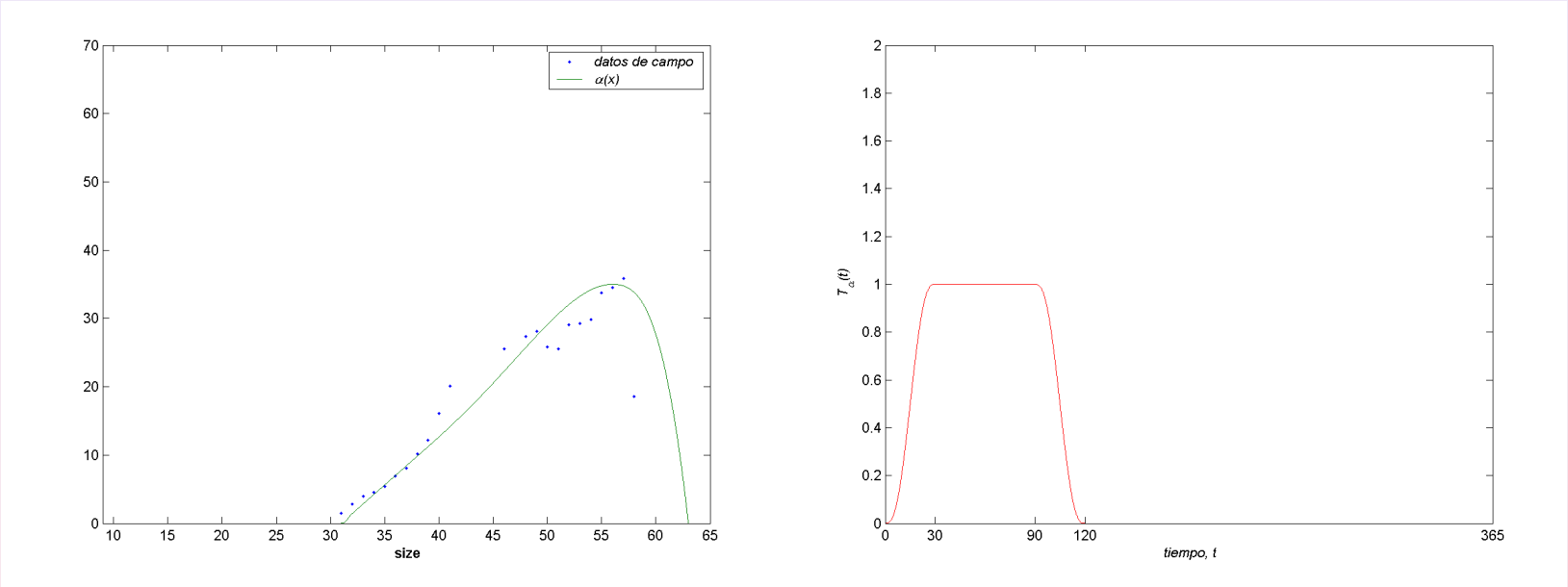
L.W. Bostford, *et al.*,
Population development of the mosquitofish, *Gambusia affinis*, in rice fields,
Environ. Biol. Fish., **20**(2)(1987) 143–154.

D. Sulsky,
Numerical solution of structured population models II. Mass structure,
J. Math. Biol., **32** (1994) 491–514.

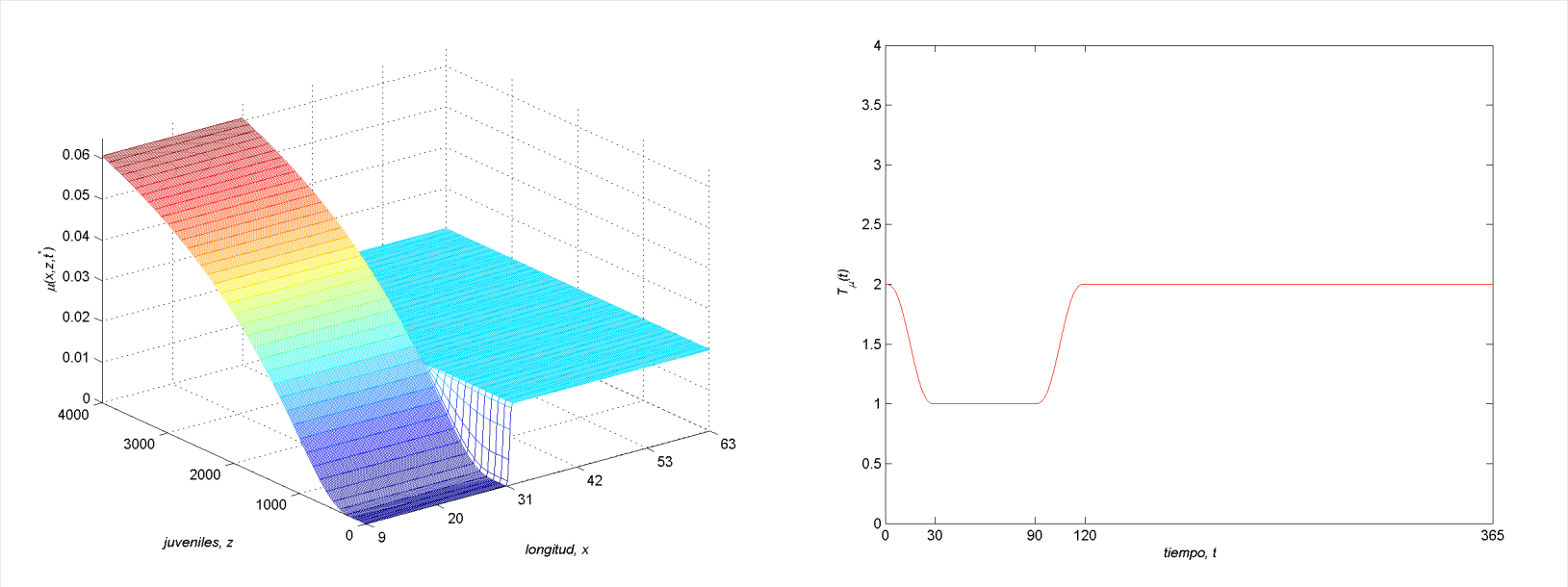
Growth rate function, $g(x,t) = g(x)\, T_g(t)$



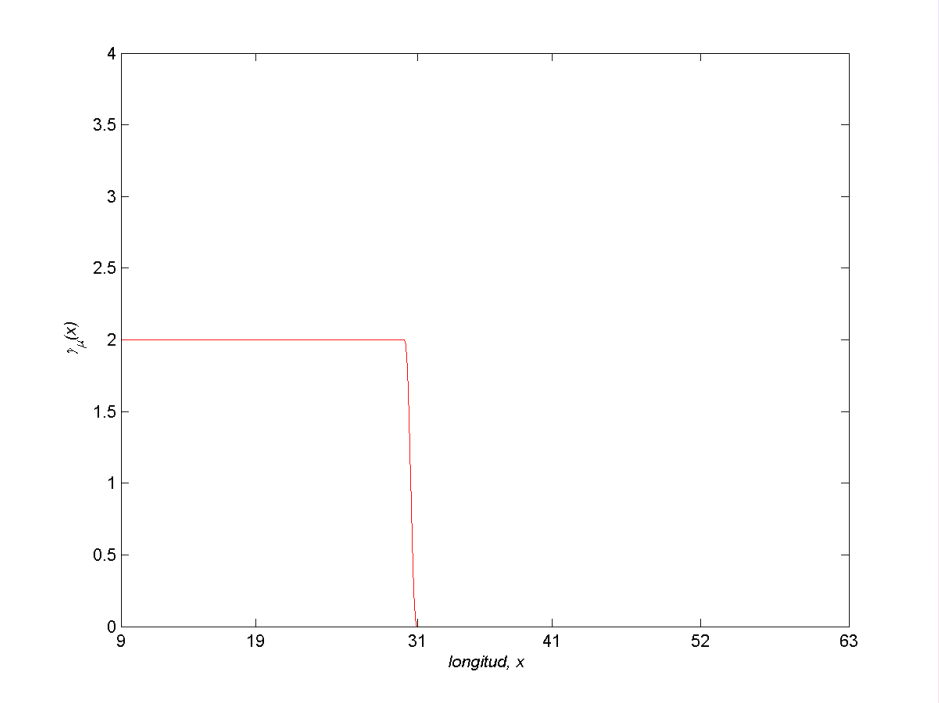
Fertility rate function, $\alpha(x,t) = \alpha(x)\, T_\alpha(t)$



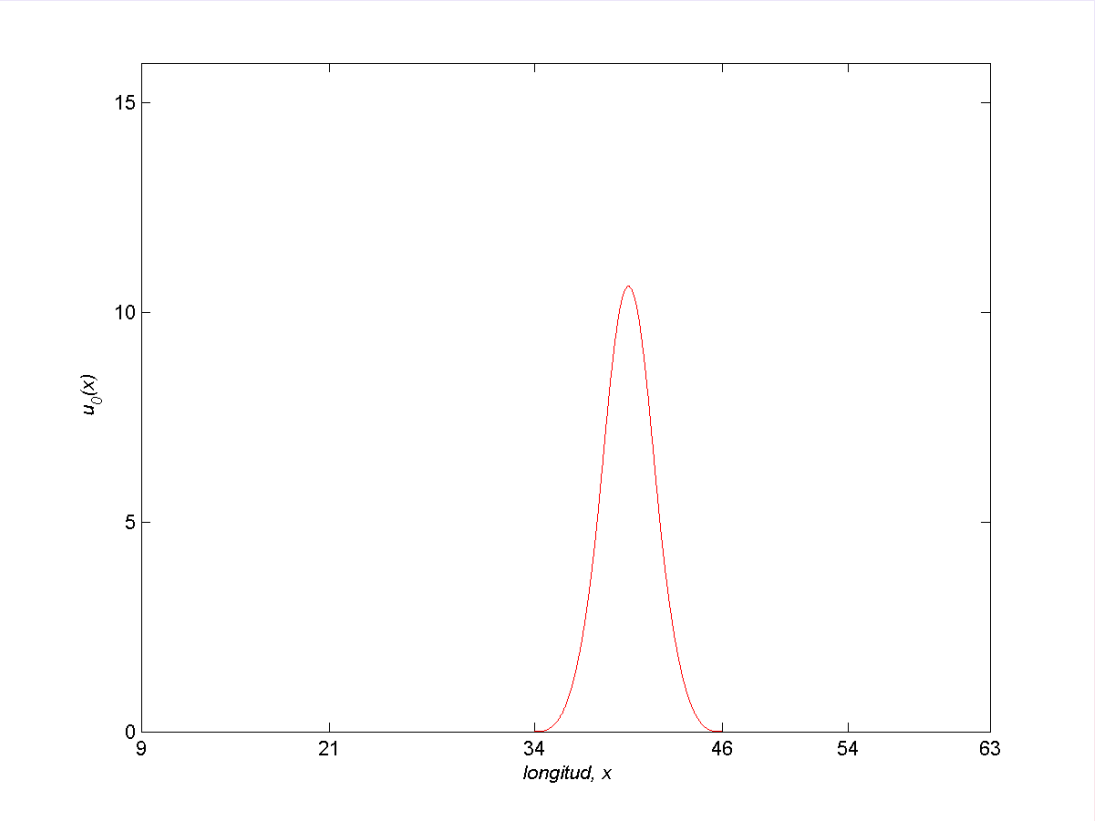
Mortality rate function, $\mu(x, z, t) = \mu(x, z)\, T_\mu(t)$



Weigth function

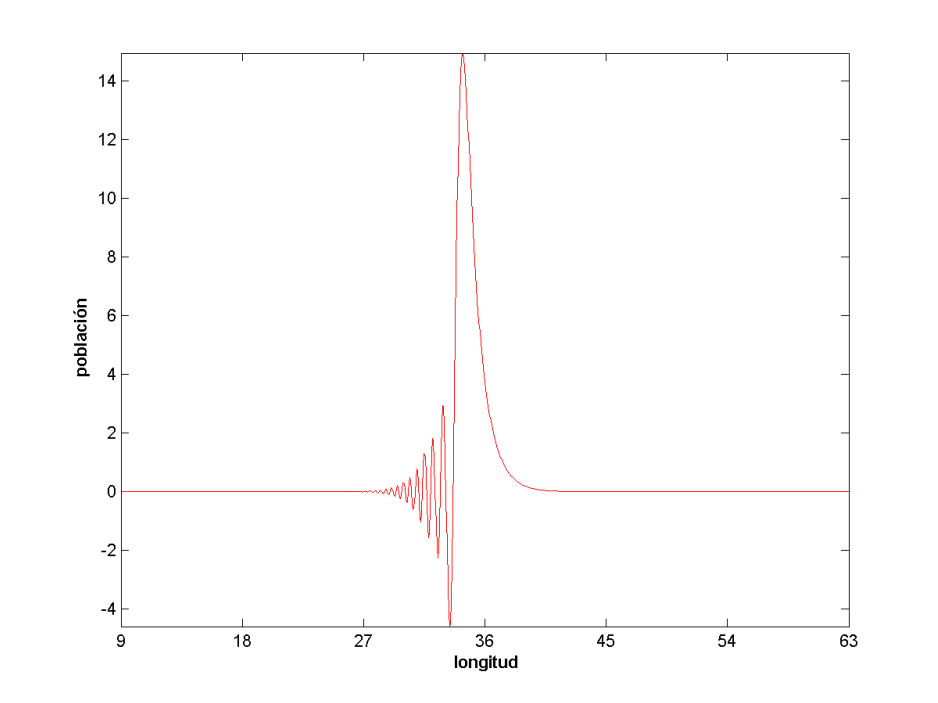


Initial condition

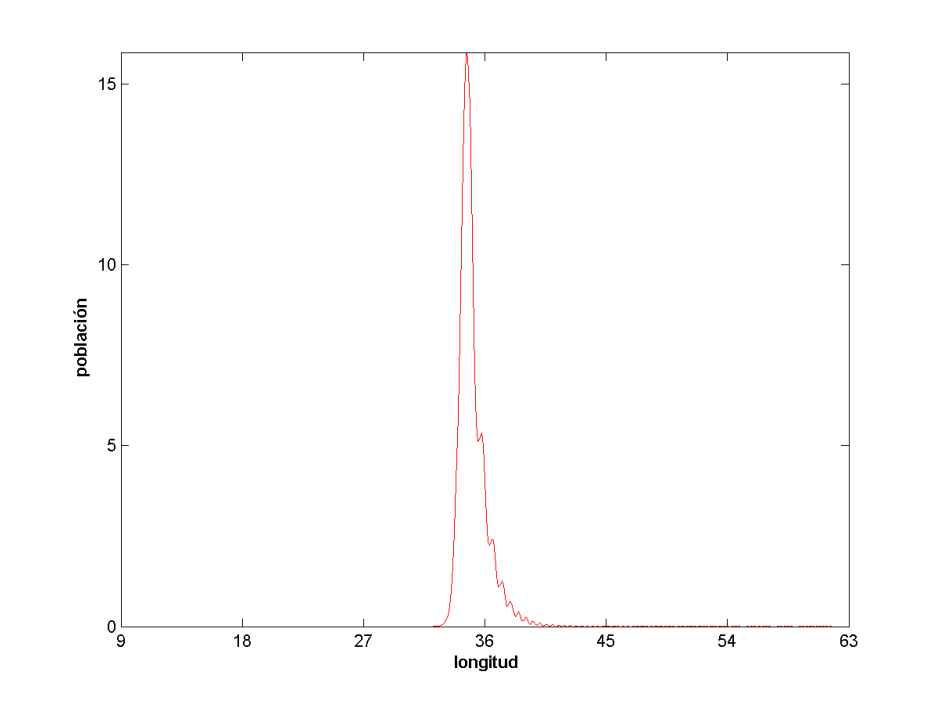


One year integration.

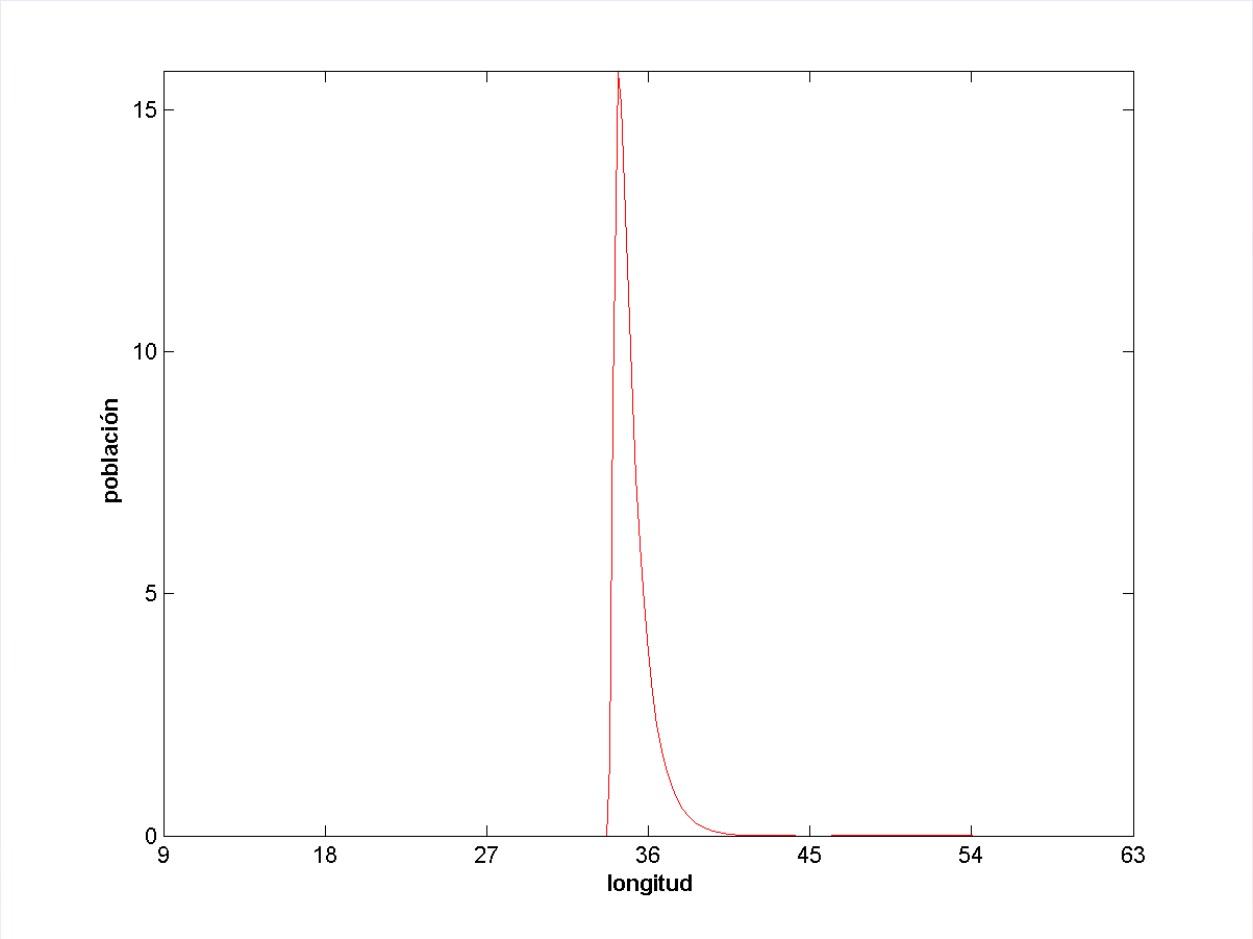
Lax-Wendroff



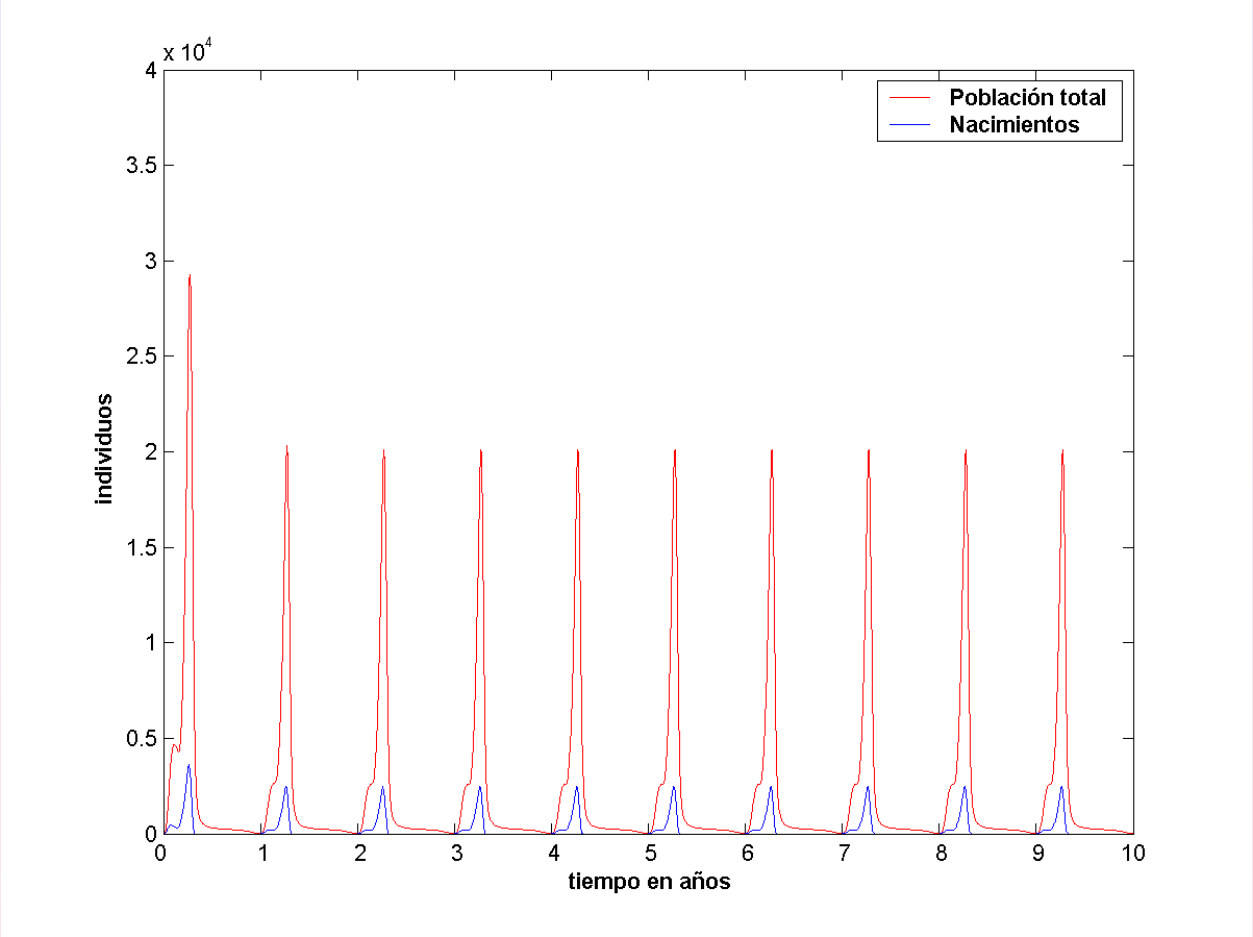
Box method



One year integration. AGN y SGN



Long time behaviour. 10 years.



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