INT. J. CONTROL, 1993, VOL. 57, NO. 3, 537–556

A Luenberger-like observer for nonlinear systems

G. CICCARELLA†, M. DALLA MORA† and A. GERMANI†

A state observer is proposed for nonlinear continuous time systems which extends the well known Luenberger observer. In particular, on the basis of simple assumptions on the regularity of the system equations (observability and the global Hölder condition for suitable functions), which are generally satisfied for physically meaningful dynamic systems, the global asymptotic convergence of the estimated state towards the true state is shown. Finally, some examples of applications are also reported showing the effectiveness of the proposed observer.

1. Introduction

The design of state estimators is one of the essential points in control theory whose solution, in the linear case, is the well known Luenberger's observer (1971).

A first systematic contribution to the theory of observers for nonlinear systems was a set of conditions under which the dynamics of the observation error is linear (Krener and Isidori 1983, Krener and Respondek 1985, Xia and Gao 1989). Unfortunately the necessary and sufficient conditions for this type of observer to exist are rather restrictive, as in the dual problem of feedback linearization.

A nice contribution to the extension of the Luenberger observer for nonlinear systems by a linearization technique has been given by Zeitz (1987). The algorithm proposed by Zeitz, which also uses time derivatives of the input, is easy to implement, but it does not in general guarantee the convergence of the observer.

A different approach based on 'high-gain' approximate cancellation of the nonlinearity has been presented by Tornambè (1989). However this approach does not guarantee, with arbitrarily high but finite gain, the asymptotic convergence of the estimated state to the true state. It is easy to demonstrate that, even if the initial state for the observer coincides with that of the system, the error could be in general only bounded and there is no guarantee that it converges asymptotically to zero.

Marino (1990) presented an adaptive observer for single-input single-output nonlinear systems that can be transformed to a certain observable canonical form, and Bastin and Gevers (1988) gave necessary and sufficient conditions for a system to be transformed into this canonical form. This observer requires a transformation which is often difficult to find and conditions given by Bastin and Gevers are restrictive from an application point of view. The proposed adaptive

Received 20 August 1990. Revised 15 December 1991 and 15 March 1992. Communicated by Professor A. Isidori.

† Department of Electrical Engineering, University of L'Aquila, Monteluco, 67100 L'Aquila, Italy.

0020-7179/93 $10.00 © 1993 Taylor & Francis Ltd

observer does not require the complete knowledge of the system model, but, on the other hand, it guarantees only an asymptotically finite error.

Tsinias (1990) presented an observer which guarantees the convergence to zero of the error. This paper provides a nice Lyapunov-like sufficient condition for the existence of a nonlinear observer. However, as sometimes is the case in similar situations, the actual construction of this Lyapunov function may be difficult.

To our knowledge, the most complete and satisfactory contribution to the theory of nonlinear observers has been recently provided by Gauthier *et al.* (1992). In this paper the authors show that, in suitable coordinates, the state of a nonlinear system can be globally asymptotically tracked by means of an observer whose 'gain' is determined via the solution of an appropriate Lyapunov-like equation. This approach essentially requires the existence of globally defined and globally lipschitzian change of coordinates — 'uniform observability'. If the system is 'uniformly observable for any input', then a globally convergent observer is found for a nonlinear system with inputs. An alternative proof of the main results given by Gauthier *et al.* (1992) has also been recently provided by Tsinias (1991).

In this paper, we present new contributions to the theory of nonlinear observers, that improve a number of the existing results summarized above. In particular:

(*a*) the construction of the observer proposed in this paper does not require a preliminary nonlinear change of coordinates;

(*b*) the calculation of the 'gain' is very straightforward;

(*c*) the observer convergence can be proved under very general conditions, which are in general satisfied for physically meaningful dynamic systems;

(*d*) it can be extended to multiple-input–multiple-output nonlinear systems;

(*e*) it is computationally simple and easily implementable.

Moreover the proposed observer, as the Luenberger-one, reproduces exactly the system equation when initialized by the true initial state.

Consider the single-input–single-output nonlinear system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t); \quad x(0) = x_0 \tag{1.1}$$

$$y(t) = h(x(t)) \tag{1.2}$$

with state $x(t) \in \mathbb{R}^n$, h a $\mathbb{C}^\infty$ real-valued function and f, $g \in \mathbb{C}^\infty$ real-valued vector fields. The class U of real-valued inputs is constituted by uniformly bounded functions, i.e. $U = \{u: |u(t)| \leqslant M, \forall t \geqslant 0\}$.

For the following we need to recall the definition of the Lie derivative $L_\varphi\lambda$ of the $\mathbb{C}^\infty$ function $\lambda(x)$ with respect to the $\mathbb{C}^\infty$ vector field φ (see Isidori, 1989):

$$L_\varphi\lambda(x) = \langle \nabla\lambda(x), \quad \varphi(x)\rangle = \sum_{i=1}^{n} \frac{\partial\lambda(x)}{\partial x_i}\, \varphi_i(x) \tag{1.3}$$

where ∇ stands for the grandient operator. Moreover the symbols $L_\varphi^k\lambda(x)$ means the k-times repeated iteration of $L_\varphi\lambda(x)$:

$$\left.\begin{aligned} L_\varphi^k\,\lambda(x) &= L_\varphi(L_\varphi^{k-1}\lambda(x))(x) \\ L_\varphi^0\,\lambda(x) &= \lambda(x) \end{aligned}\right\} \tag{1.4}$$

In what follows, we also denote by $Q(x)$ the so-called observability matrix of (1.1), (1.2) (see Isidori 1989):

$$Q(x) = \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \tag{1.5}$$

In the linear case, i.e. $h(x) = Cx$ and $f(x) = Ax$, the matrix (1.5) reduces to the well known observability matrix.

Moreover by the symbol γ_Q and δ we mean the Hölder constants associated with Q, i.e.

$$\|Q(x) - Q(y)\| \leq \gamma_Q \|x - y\|^\delta$$

with $\delta \in (0, 1]$. Of course the case $\delta = 1$ corresponds to the usual Lipschitz condition. Finally, for a bounded set $\Gamma \in \mathbb{R}^n$ we let radius (Γ) denote the quantity:

$$\text{radius } (\Gamma) = \sup_{x \in \Gamma} \|x\|$$

2. The observer for nonlinear systems

This section presents the equation for the observer for single-input–single-output continuous time nonlinear systems by considering first the case of systems without input and with scalar output. The case of systems with input will be presented in § 3.

In order to prove the main result of this section we need to state the following lemmas.

Lemma 1: *Let $V(\lambda)$ be the Vandermonde matrix*

$$V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{bmatrix}$$

then for any $\alpha > 0$ and $c > 0$ there exist $\lambda_n < \lambda_{n-1} < \dots < \lambda_1 < 0$ such that the following equation is satisfied:

$$\lambda_1 + \|V^{-1}(\lambda)\| c = -\alpha \tag{2.1}$$

Proof: Let

$$y = F(\lambda) = \lambda_1 + \|V^{-1}(\lambda)\| c + \alpha$$

and observe that there exist λ' and λ'' such that

(i) $$F(\lambda') > 0$$

(ii) $$F(\lambda'') < 0$$

For (*i*) it is enough to observe that $V(\lambda)$ becomes singular when $|\lambda_i - \lambda_j| \to 0$, $i \neq j$, so that $\|V^{-1}(\lambda)\| \to +\infty$.

On the other hand (ii) can be obtained by noting that, choosing for example $\bar{\lambda}_i = -w^i$, $i = 1, 2, \dots, n$, we have:

$$\lim_{w \to \infty} \|V^{-1}(-w, -w^2, \dots, -w^n)\| = 1 \tag{2.2}$$

which implies $\lim_{w\to\infty} F(\tilde{\lambda}) = -\infty$, so that, by a classical continuity argument, there exists at least a finite solution of $F(\lambda) = 0$. □

Lemma 2: *Let $m(t)$ be a non-negative scalar function such that for $\delta \in (0, 1)$, $\eta > 0$*

$$m(t) \leqslant e^{\lambda t} m(0) + \eta \int_0^t e^{\lambda(t-\tau)} [m(\tau)]^\delta \, d\tau \tag{2.3}$$

then

$$m(t) \leqslant m(0) \left[e^{\lambda(1-\delta)t} + \frac{\eta}{\lambda} m(0)^{(\delta-1)} (e^{\lambda(1-\delta)t} - 1) \right]^{1/(1-\delta)} \tag{2.4}$$

Proof: Let $r(t)$ be the solution of the following integral equation:

$$r(t) = e^{\lambda t} r(0) + \eta \int_0^t e^{\lambda(t-\tau)} [r(\tau)]^\delta \, d\tau$$

it is easy to verify that

$$r(t) = r(0) \left[e^{\lambda(1-\delta)t} + \frac{\eta}{\lambda} r(0)^{(\delta-1)} (e^{\lambda(1-\delta)t} - 1) \right]^{1/(1-\delta)}$$

now, by using a generalized version of the Gronwall inequality (Lakshmikantham *et al.* 1989), it follows that if $m(0) \leqslant r(0)$, then $m(t) \leqslant r(t)$, for all $t \geqslant 0$, which proves (2.4). □

Remark 1: Note that when $\delta \to 1$, (2.4) becomes the well known Gronwall inequality:

$$m(t) \leqslant e^{(\lambda+\eta)t} m(0) \tag{2.5}$$

which can easily be derived directly from (2.3). □

Theorem 1: *Let $Q(x)$ be the observability matrix associated with the pair $(f(x), h(x))$ in (1.1), (1.2). Assume $u(t) = 0$, for all $t \geqslant 0$, if:*

(H1) *$Q(x)$ has full rank for all $x \in \mathbb{R}^n$*

(H2) *$L_f^n h(\phi^{-1}(\zeta))$ is uniformly Hölder, i.e. for all $\zeta_1, \zeta_2 \in \mathbb{R}^n$:*

$$\|L_f^n h(\phi^{-1}(\zeta_1)) - L_f^n h(\phi^{-1}(\zeta_2))\| \leqslant \gamma \|\zeta_1 - \zeta_2\|^\delta$$

with $\delta \in (0, 1]$

then there exists a finite gain vector $K \in \mathbb{R}^n$ such that the solution of the following system equations:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t)) + [Q(\hat{x}(t)]^{-1} K[y(t) - h(\hat{x}(t))] \\ \hat{x}(0) &= \bar{x} \in \mathbb{R}^n \end{aligned}\right\} \tag{2.6}$$

has the following properties:

(i) *for $\delta \in (0, 1)$,*

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| \leqslant \varepsilon$$

for any $\varepsilon > 0$, for all $\hat{x}(0) \in \mathbb{R}^n$.

(ii) *for* $\delta = 1$

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0, \quad \forall\, \hat{x}(0) \in \mathbb{R}^n$$

Proof: Let us define the following nonlinear transformation:

$$z = \Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}, \quad x \in \Omega \subset \mathbb{R}^n \tag{2.7}$$

which admits the inverse because of the implicit function theorem and H1:

$$x = \Phi^{-1}(z) \tag{2.8}$$

In the new coordinate system the state equations (1.1) and (1.2) become:

$$\dot{z}(t) = Az(t) + BL_f^n h(\Phi^{-1}(z(t))) \tag{2.9}$$

$$y(t) = Cz(t) \tag{2.10}$$

where A $(n \times n)$, B $(n \times 1)$ and C $(1 \times n)$ are given by the Brunowsky canonical form (1970):

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}; \quad C = [1 \quad 0 \quad \dots \quad 0]$$

In the same way, defining $\hat{z}(t) = \Phi(\hat{x}(t))$ the equation (2.6) for the observer assumes the form:

$$\left.\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + BL_f^n h[\Phi^{-1}(\hat{z}(t))] + K\,[y(t) - C\hat{z}(t)] \\ &= (A - KC)\hat{z}(t) + BL_f^n h[\Phi^{-1}(\hat{z}(t))] + Ky(t) \\ \hat{z}(0) &= \Phi(\bar{x}) \end{aligned}\right\} \tag{2.11}$$

Now we study the properties of the estimation error $\xi(t)$ in the transformed state

$$\xi(t) = \hat{z}(t) - z(t)$$

By (2.9) and (2.11) we can write

$$\dot{\xi}(t) = (A - KC)\xi(t) + B\,[L_f^n h(\Phi^{-1}(\hat{z}(t))) - L_f^n h(\Phi^{-1}(z(t)))] \tag{2.12}$$

The companion matrix $(A - KC)$ can be diagonalized by the Vandermonde transformation

$$V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{bmatrix}$$

where $\lambda = [\lambda_1, \dots, \lambda_n]$ is the vector of the eigenvalues of $(A - KC)$, that, due to the observability property of (A, C), can be assigned with K.

Moreover, for the sake of simplicity, we can assume the λ_i to be real and

$\lambda_n < \lambda_{n-1} < \ldots < \lambda_1 < 0$. Then

$$e^{(\Lambda - KC)t} = V^{-1}(\lambda)e^{\Lambda t}V(\lambda) \tag{2.13}$$

where $\Lambda = \mathrm{diag}[\lambda_1 \ \ldots \ \lambda_n]$.

From (2.12) we have:

$$V(\lambda)\xi(t) = e^{\Lambda t}V(\lambda)\xi(0) + \int_0^t e^{\Lambda(t-\tau)}V(\lambda)B\,[L_f^n h(\Phi^{-1}(\hat{z}(\tau))) - L_f^n h(\Phi^{-1}(z(\tau)))]\,d\tau \tag{2.14}$$

Taking the norm, observing that $\|V(\lambda)\ B\| = \sqrt{n}$, and, considering H2, we have

$$\|V(\lambda)\xi(t)\| \leq \|V(\lambda)\xi(0)\|\,e^{\lambda_1 t} + \int_0^t e^{\lambda_1(t-\tau)}\sqrt{n}\,\gamma\|\,V^{-1}(\lambda)\|^{\delta}\,\|V(\lambda)\xi(\tau)\|^{\delta}\,d\tau \tag{2.15}$$

By Lemma 2 we obtain

$$\|V(\lambda)\xi(t)\| \leq \|V(\lambda)\xi(0)\|\left\{e^{\lambda_1(1-\delta)t} + \frac{\sqrt{n}\,\gamma}{\lambda_1}\|V^{-1}(\lambda)\|^{\delta}\,\|V(\lambda)\xi(0)\|^{(\delta-1)}(e^{\lambda_1(1-\delta)t} - 1)\right\}^{1/(1-\delta)} \tag{2.16}$$

With reference to case (i) the asymptotic error is

$$\lim_{t\to\infty}\|V(\lambda)\xi(t)\| \leq \left(\frac{\sqrt{n}\,\gamma\|V^{-1}(\lambda)\|^{\delta}}{|\lambda_1|}\right)^{1/(1-\delta)}$$

which implies

$$\lim_{t\to\infty}\|\xi(t)\| \leq \left(\frac{\sqrt{n}\,\gamma\|V^{-1}(\lambda)\|}{|\lambda_1|}\right)^{1/(1-\delta)}$$

By Lemma 1 this asymptotic upper bound for the error can be chosen arbitrarily small. Invoking the continuity of Φ^{-1}, the thesis is proved.

With reference to case (ii), by the Remark 1 to Lemma 2 we obtain:

$$\begin{aligned}\|V(\lambda)\xi(t)\| &\leq e^{(\lambda_1 + \|V^{-1}(\lambda)\|\sqrt{n}\gamma)t}\,\|V(\lambda)\xi(0)\| \\ &= e^{-\alpha t}\|V(\lambda)\xi(0)\|\end{aligned} \tag{2.17}$$

which, by continuity of Φ^{-1}, implies the thesis as $\lambda_1, \ldots, \lambda_n$ can be chosen according to Lemma 1. □

Remark 2: Note that when $\delta = 1$ the convergence of $\hat{z}(t)$ to $z(t)$ has already been proved by Gauthier *et al.* (1992). They used a gain K that depends only on one parameter and is determined via the solution of a Lyapunov-like equation. In our proof, the possibility of choosing $\lambda_1, \ldots, \lambda_n$ provides a greater number of degrees of freedom. This allows in principle the optimization of the behaviour of the observer by a suitable choice of the eigenvalues for any physical application. Moreover, the case $\delta < 1$ is also covered. To conclude, we would like to stress that also in the case of simple systems the nonlinear term

$L_f^n h(\phi^{-1}(z))$, which is used in the observer equations given by Gauthier *et al.* (1992), is in general very difficult to compute. □

Remark 3: Note also that when Φ^{-1} is uniformly Lipschitz then in case (ii) the convergence of $\hat{x}$ to x is exponential. In fact, let $\gamma_{\Phi^{-1}}$ be the Lipschitz constant associated with Φ^{-1}; we then have

$$\begin{aligned}\|\hat{x}(t) - x(t)\| &= \|\Phi^{-1}(\hat{z}(t)) - \Phi^{-1}(z(t))\| \leqslant \gamma_{\Phi^{-1}} \|\hat{z}(t) - z(t)\| \\ &\leqslant \gamma_{\Phi^{-1}}\|V^{-1}(\lambda)\|\|V(\lambda)\xi(t)\| \leqslant \gamma_{\Phi^{-1}}\|V^{-1}(\lambda)\|e^{-\alpha t}\|V(\lambda)\xi(0)\|\end{aligned}$$

□

Remark 4: We would like to observe that when $\delta > 1$ (the case of 'low gain'), the quantity $L_f^n h(\Phi^{-1}(z))$ is independent of z and therefore any choice of $0 > \lambda_1 > \ldots > \lambda_n$ guarantees the local convergence of $\hat{x}(t)$ to $x(t)$. Then for 'small' eigenvalues, small gain K can be used. When $\delta = 1$ (the case of 'finite gain') H2 becomes the uniform Lipschitz condition and the convergence to zero of the error is guaranteed for any choice of $\hat{x}(0)$ if the finite gain K is suitably chosen in order to satisfy the condition

$$\lambda_1 + \sqrt{n}\,\gamma\|V^{-1}(\lambda)\| < 0$$

Finally, when $\delta \in (0, 1)$ (the case of 'high gain'), the asymptotic error is smaller than a positive constant which can be made as small as desired by taking $\|K\|$ sufficiently high. □

Remark 5: For the practical computation of K it is enough to choose $\text{Real}\{\lambda_i\} < -\gamma$, for all $i = 1, 2, \ldots, n$ and to compute the coefficients of the polynomial

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n) = \lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + \ldots + K_n$$

Then $K = [K_1 \;\; K_2 \;\; \ldots \;\; K_n]^{\mathrm{T}}$. □

The hypotheses H1 and H2 are considerably weakened when it is known *a priori* that the evolution of the system state $x(t)$ is confined to a bounded closed sphere $S(\rho)$ of a given radius ρ, around the origin in $\mathbb{R}^n$.

In this case we have the following result.

Corollary 1: *The estimation error for the observer* (2.6) *converges exponentially to zero if*

(i) $L_f^n h(\Phi^{-1}(\xi))$ *is locally Lipschitz*;

(ii) $x(t) \in S(\rho)$, for all $t \geqslant 0$;

(iii) $Q(x)$ *has full rank for all* $x \in S$ *(radius* $\{\Phi^{-1}(S(Nr + r))\}$*) for an arbitrarily fixed* $N > 1$ *and* r *is such that* $\|\Phi(x)\| \leqslant r$, for all $x \in S(\rho)$;

(iv) $\|\hat{x}(0) - x(0)\| \leqslant \psi$, *for a suitable* $\psi > 0$.

Proof: It is sufficient to observe that H1 and H2 are verified in the bounded set S (radius $\{\Phi^{-1}(S(Nr + r))\}$), because if a function is locally Lipschitz then it is uniformly Lipschitz in any compact set, and to show that

$$\hat{x}(t) - x(t) \in S \text{ (radius } \{\Phi^{-1}(S(Nr + r))\}), \quad \forall\, t \geqslant 0$$

Without loss of generality, it is possible to choose

$$\lambda = (\lambda_1, \ldots, \lambda_n)^{\mathrm{T}}$$

in such a way that $\|V^{-1}(\lambda)\| \leq N$, with the fixed $N > 1$ and that

$$\lambda_1 + \|V^{-1}(\lambda)\| \sqrt{n}\,\gamma = -\alpha \tag{2.18}$$

is satisfied.

In fact, for any choice of $\lambda_1 < 0$ choosing $\lambda = \lambda - \varepsilon^i$, $\varepsilon > 0$ we have

$$\lim_{\varepsilon \to 0} \|V^{-1}(\lambda)\| = \infty$$

$$\lim_{\varepsilon \to \infty} \|V^{-1}(\lambda)\| = 1$$

so that there exists $\varepsilon_{N,\lambda_1} > 0$ such that

$$1 \leq \|V^{-1}(\lambda)\| \leq N, \quad \forall\, \varepsilon > \varepsilon_{N,\lambda_1} \tag{2.19}$$

Moreover, for such a choice of the λ_i a continuous function $m(\lambda_1, \varepsilon) = \|V^{-1}(\lambda)\|$ is well defined. This function is such that for all $\beta \in (1, N]$, $\exists\, \bar{\varepsilon} > \varepsilon_{N,\lambda_1}$: $m(\lambda_1, \bar{\varepsilon}) = \beta$ such that (2.18) can be satisfied by choosing any

$$-\lambda_1 \in (\alpha + \sqrt{n}\,\gamma,\ \alpha + \sqrt{n}\,\gamma N]$$

and $\bar{\varepsilon}$ such that

$$m(\lambda_1, \bar{\varepsilon}) = (-\alpha - \lambda_1)/\sqrt{n}\,\gamma$$

As $x(t) \in S(\rho)$ it follows that $\Phi(x(t)) \subset S(r)$ for a suitable choice of r according to the hypothesis (iii).

Let us now define the set $U_\lambda(r) \subset \mathbb{R}^n$

$$U_\lambda(r) = \{\xi \in \mathbb{R}^n \colon \|V(\lambda)\,\xi\| \leq r\}$$

and observe that from (2.19), denoting

$$\lambda_m = [\text{minimum eigenvalue } V^{\mathrm{T}}(\lambda)V(\lambda)]^{1/2}$$

we have:

$$\frac{1}{N} \leq \lambda_m \leq 1 \tag{2.20}$$

In fact

$$\lambda_m^2 = \min_{x \in \mathbb{R}^n} \frac{x^{\mathrm{T}}(V^{\mathrm{T}}(\lambda)V(\lambda))x}{x^{\mathrm{T}}x}$$

$$= \frac{1}{\max\limits_{y} \dfrac{y^{\mathrm{T}}(V^{-\mathrm{T}}(\lambda)V^{-1}(\lambda))y}{y^{\mathrm{T}}y}} = \frac{1}{\|V^{-1}(\lambda)\|^2} \tag{2.21}$$

which implies (2.20).

$$\begin{aligned} U_\lambda(r) &= \{\xi\colon \|V(\lambda)\xi\| \leqslant r\} \\ &= \{\xi\colon \xi^{\mathrm{T}} V^{\mathrm{T}}(\lambda) V(\lambda)\, \xi \leqslant r^2\} \\ &\subset \{\xi \in \mathbb{R}^n\colon \lambda_m^2 \|\xi\|^2 \leqslant r^2\} \\ &\subset \{\xi \in \mathbb{R}^n\colon \|\xi\| \leqslant Nr\} = S(Nr) \end{aligned} \tag{2.22}$$

From (2.22) and (2.17), if $\xi(0) \in U_\lambda(r)$ then $\xi(t) \in U_\lambda(r)$, i.e.

$$\xi(t) \in S(Nr) \quad \forall t \geqslant 0$$

Consequently $z(t)$ and $\hat{z}(t)$ belong to $S(Nr + r)$, which guarantees the existence of the Lipschitz constant γ on the bounded set $S(Nr + r)$ and $\hat{x}(t)$, $x(t) \in S$ (radius $\{\Phi^{-1}(S(Nr + r))\}$).

Moreover $\xi(0) \in U_\lambda(r)$ means

$$\|V(\lambda)(\Phi(\hat{x}(0)) - \Phi(x(0)))\| \leqslant r$$

which is implied by

$$\|(\Phi(\hat{x}(0)) - \Phi(x(0)))\| \leqslant \frac{r}{\|V(\lambda)\|}$$

and then, by continuity, there exists $\psi(r/\|V(\lambda)\|)$ such that $\xi(0) \in U_\lambda(r)$ if

$$\|\hat{x}(0) - x(0)\| \leqslant \psi\left(\frac{r}{\|V(\lambda)\|}\right) \qquad \square$$

Remark 6: This case has also been considered by Gauthier *et al.* (1992) by defining, in our terminology, a function $\varphi(z)$ which corresponds to $L_f^n h(\Phi^{-1}(z))$ in the domain of the state evolution $S(\rho)$ and any fixed uniformly Lipschitz function outside $S(\rho)$. This does not allow a simple description of the observer in the original state, while our result does. $\square$

3. The observer for the forced system

In this section, we consider systems described by the equations (1.1), (1.2) where the input is assumed to be uniformly bounded. The observer for this class of systems can be defined by using geometrical or metrical properties for the vector fields which define the system. The following theorem shows the convergence of an observer derived from (2.6) for the class of systems having relative degree equal to n.

Theorem 2: *Let $Q(x)$ be the observability matrix associated with the pair $(f(x), h(x))$ of functions in* (1.1), (1.2). *Assume $u(t) \in U$, for all $t \geqslant 0$, and the triple $(f(x), g(x), h(x))$ has relative degree n (see Isidori* 1989) *if*

(H1) *$Q(x)$ has full rank for all $x \in \mathbb{R}^n$; and*

(H2)

$$\begin{aligned} \sup_{\|u\| \leqslant M} \|L_f^n h(\phi^{-1}(\zeta_1)) &+ uL_g L_f^{n-1} h(\phi^{-1}(\zeta_1)) - L_f^n h(\phi^{-1}(\zeta_2)) \\ &- uL_g L_f^{n-1} h(\phi^{-1}(\zeta_2))\| \leqslant \gamma_g \|\zeta_1 - \zeta_2\|^\delta \end{aligned}$$

with $\delta \in (0, 1]$, $\forall\, \zeta_1, \zeta_2 \in \mathbb{R}^n$

then there exists a finite gain vector $K \in \mathbb{R}^n$ such that the solution of the following system equations:

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + [Q(\hat{x}(t))]^{-1}K\,[y(t) - h(\hat{x}(t))] \tag{3.1}$$
$$\hat{x}(0) = \bar{x}$$

has the following properties:

(i) *for* $\delta \in (0, 1)$

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| \leq \varepsilon \tag{3.2}$$

for any $\varepsilon > 0$, *for all* $\hat{x}(0) \in \mathbb{R}^n$; *and*

(ii) for $\delta = 1$

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0 \quad \forall\, \hat{x}(0) \in \mathbb{R}^n$$

Proof: Using the same nonlinear transformation (2.7), (2.8) we can write

$$\dot{z}(t) = (A - KC)z(t) + B[L_f^n h(\phi^{-1}(z(t))) + L_g L_f^{n-1} h(\phi^{-1}(z(t)))u(t)] + Ky(t) \tag{3.3}$$

$$\dot{\hat{z}}(t) = (A - KC)\hat{z}(t) + B[L_f^n h(\phi^{-1}(\hat{z}(t))) + L_g L_f^{n-1} h(\phi^{-1}(\hat{z}(t)))u(t)] + Ky(t) \tag{3.4}$$

The error as in Theorem 1 is

$$\|V(\lambda)\xi(t)\| \leq \|V(\lambda)\xi(0)\| \left(e^{\lambda_1(1-\delta)t} + \frac{\sqrt{n}}{\lambda_1}\gamma_g \|V^{-1}(\lambda)\|^{\delta}\, \|V(\lambda)\xi(0)\|^{(\delta-1)}(e^{\lambda_1(1-\delta)t} - 1)\right)^{1/(1-\delta)} \tag{3.5}$$

Then the theorem is proved following the same method as for Theorem 1. □

Also in this case we can state the analogous of Corollary 1.

Corollary 2: *The estimation error for the observer* (3.1) *converges exponentially to zero if*

(i) $L_f^n h(\phi^{-1}(\zeta))$ *and* $L_g L_f^{n-1} h(\phi^{-1}(\zeta))$ *are locally Lipschitz;*

(ii) *there exists a closed bounded sphere* $S(\rho)$ *of radius* ρ *such that* $x(t) \in S(\rho)$, $\forall u \in U$

(iii) $Q(x)$ *has full rank for all* $x \in S$ *(radius* $\{\Phi^{-1}(S(Nr + r))\}$*) where* $N > 1$ *is an arbitrarily fixed constant and* r *is such that* $\|\Phi(x)\| \leq r$, *for all* $x \in S(\rho)$; and

(iv) $\|\hat{x}(0) - x(0)\| \leq \psi$, *for a suitable* $\psi > 0$.

Proof: The proof is exactly as in Corollary 1, observing that also the Lipschitz property for $L_g L_f^{n-1} h(\phi^{-1}(\zeta))$ is guaranteed in S (radius $\{\Phi^{-1}(S(Nr + r))\}$). □

For nonlinear systems which do not have relative degree n, it is still possible to prove the asymptotic convergence of the proposed observer.

In this case, we need the hypothesis that the class of admissible inputs U is given by piecewise polynomial functions of order r almost everywhere in t, i.e. $u \in U^{(r)}$ if

$$\frac{\mathrm{d}^{r+1}u(t)}{\mathrm{d}t^{r+1}} = u^{(r+1)}(t) = 0, \quad \text{a.e. for } t \geqslant 0$$

Note that such a class is dense in the set of continuous functions, it contains spline functions and therefore it is not really a constraint for physical applications.

For the next theorem we need the following definitions. Let $m(k) = \min(k-3, r-1)$

$$\begin{aligned}
D &= [u^{(0)}(t)\ u^{(1)}(t) \ldots u^{(r)}(t)]^{\mathrm{T}} \\
\psi_1(x, D) &= h(x) \\
\psi_2(x, D) &= L_{f+gu}h(x) \\
&\vdots \\
\psi_k(x, D) &= L_{f+gu}\psi_{k-1}(x, D) + \sum_{i=0}^{m(k)} \frac{\mathrm{d}}{\mathrm{d}u^{(i)}}\, \psi_{k-1}(x, D)\, u^{(i+1)} \\
\Phi(x, D) &= [\psi_1(x, D) \quad \psi_2(x, D) \ldots \psi_n(x, D)]^{\mathrm{T}}
\end{aligned}$$

and, when it exists, we will denote by $\Phi^{-1}(\zeta, D)$ the solution in x of the equation

$$\Phi(x, D) = \zeta$$

Now we can state the following theorem.

Theorem 3: *Assume $u(t) \in U^{(r)}$ for all $t \geqslant 0$. If*

(H1) *$Q(x, D) = \mathrm{d}\Phi(x, D)/\mathrm{d}x$ has full rank for all $x \in \mathbb{R}^n$, for all $D \in \mathbb{R}^{r+1}$ and*

(H2) *$\psi_{n+1}(\Phi^{-1}(\zeta, D), D)$ is uniformly Hölder, i.e. there exists $\delta \in (0, 1]$ such that*

$$|\psi_{n+1}(\Phi^{-1}(\zeta, D), D) - \psi_{n+1}(\Phi^{-1}(\bar{\zeta}, \bar{D}), \bar{D})| \leqslant \gamma \left\| \begin{matrix} \zeta - \bar{\zeta} \\ D - \bar{D} \end{matrix} \right\|^{\delta}$$

then, denoting by A a Brunowsky matrix of dimension $r+1$, there exist finite gain vectors K_1 and K_2 such that the following system equations for the estimation of D and x:

$$\begin{aligned}
\dot{\hat{D}}(t) &= A\hat{D}(t) + K_2(u(t) - D_1(t)) \\
\dot{\hat{x}}(t) &= f(\hat{x}(t)) + g(\hat{x}(t))\hat{D}_1(t) + \left(\frac{\mathrm{d}\Phi(\hat{x}, \hat{D})}{\mathrm{d}\hat{x}}\right)^{-1} K_1[y(t) - h(\hat{x}(t))] \\
&\quad - \left(\frac{\mathrm{d}\Phi(\hat{x}, \hat{D})}{\mathrm{d}\hat{x}}\right)^{-1} \frac{\mathrm{d}\Phi(\hat{x}, \hat{D})}{\mathrm{d}\hat{D}}\, K_2[u(t) - \hat{D}_1(t)]
\end{aligned} \tag{3.6}$$

have the following properties:

(i) *for* $\delta \in (0, 1)$,

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| \leq \varepsilon$$

for any $\varepsilon > 0$, *for all* $\hat{x}(0) \in \mathbb{R}^n$; and

(ii) *for* $\delta = 1$

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0 \quad \forall\, \hat{x}(0) \in \mathbb{R}^n$$

Proof: Let us rewrite the system of equations (1.1), (1.2) in the form

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))D_1(t) \\ \dot{D}_1(t) &= D_2(t) \\ &\vdots \\ \dot{D}_r(t) &= D_{r+1} \\ \dot{D}_{r+1}(t) &= 0, \qquad \text{a.e.} \end{aligned}\right\} \tag{3.7}$$

$$\left.\begin{aligned} y(t) &= h(x(t)) \\ u(t) &= D_1(t) \end{aligned}\right\} \tag{3.8}$$

and let us define

$$w = \begin{bmatrix} x \\ \hline D_1 \\ \vdots \\ D_{r+1} \end{bmatrix} \tag{3.9}$$

$$F(w) = \begin{bmatrix} f(x) + g(x)D_1 \\ D_2 \\ \vdots \\ D_{r+1} \\ 0 \end{bmatrix}$$

$$H(w) = h(x)$$

so that (3.7) and (3.8) can be rewritten

$$\left.\begin{aligned} \dot{w} &= F(w) \\ y &= H(w) \end{aligned}\right\} \tag{3.10}$$

Moreover let us define the following nonlinear transfer motion:

$$\begin{bmatrix} z_1 \\ \hline z_2 \end{bmatrix} = \Theta(w) = \begin{bmatrix} \Theta_1(w) \\ \hline \Theta_2(w) \end{bmatrix} = \begin{bmatrix} H(w) \\ L_F H(w) \\ \vdots \\ L_F^{n-1} H(w) \\ \hline w_{n+1} \\ \vdots \\ w_{n+r+1} \end{bmatrix} \tag{3.11}$$

$$\Theta_1(w) \in \mathbb{R}^n; \quad \Theta_2(w) \in \mathbb{R}^{r+1}$$

so that (3.7) and (3.8) become

$$\dot{z}(t) = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] z(t) + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] L_F^n H(\Theta^{-1}(z(t)))$$

$$y(t) = C_1 z = [1\ 0\ \ldots\ 0]\ z$$

$$u(t) = C_2 z = [\underbrace{0\ \ldots\ 0}_{n \text{ times}}\ 1\ 0\ \ldots\ 0]\ z$$

with A_1 and A_2 Brunowsky matrices of dimension n and $r+1$ respectively and B as in Theorem 1. Moreover it is easily noted that

$$L_F^n H(\Theta^{-1}(z(t))) = \psi_{n+1}(\phi^{-1}(z_1, z_2), z_2)$$

and that the couple

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \quad \text{and} \quad C = \left[\begin{array}{c} C_1 \\ \hline C_2 \end{array}\right]$$

is observable. Then defining $\hat{w}$ as $[\hat{x}\ \hat{D}_1\ \ldots\ \hat{D}_{r+1}]^{\mathrm{T}}$ the same nonlinear transformation (3.11) gives:

$$\hat{z} = \Theta(\hat{w})$$

and

$$\dot{\hat{z}}(t) = \left[\begin{array}{c|c} A_1 - K_1C_1 & 0 \\ \hline 0 & A_2 - K_2C_2 \end{array}\right] \hat{z}(t) + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] \psi_{n+1}(\phi^{-1}(z_1, z_2), z_2)$$
$$+ \left[\begin{array}{c|c} K_1 & 0 \\ \hline 0 & K_2 \end{array}\right] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$$

The estimation error in the transformed state is given by $\xi(t) = \hat{z}(t) - z(t)$

$$\dot{\xi}(t) = \left[\begin{array}{c|c} A_1 - K_1C_1 & 0 \\ \hline 0 & A_2 - K_2C_2 \end{array}\right] \xi(t) + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] [\psi_{n+1}(\phi^{-1}(\hat{z}_1, \hat{z}_2), \hat{z}_2)$$
$$- \psi_{n+1}(\phi^{-1}(z_1, z_2), z_2)]$$

following the same line as in Theorem 1, denoting by

$$V(\lambda) = \left[\begin{array}{c|c} V_1(\lambda) & 0 \\ \hline 0 & V_2(\lambda) \end{array}\right]$$

with $V_i(\lambda)$ the Vandermonde matrices associate with $A_i - K_iC_i$, $i = 1, 2$, we obtain, as in Theorem 1,

$$\|V(\lambda)\xi(t)\| \leqslant \|V(\lambda)\xi(0)\|\, e^{\lambda_1 t} + \int_0^t e^{\lambda_1(t-\tau)} \sqrt{n}\,\gamma \|V^{-1}(\lambda)\|^{\delta}\, \|V(\lambda)\xi(\tau)\|^{\delta}\, d\tau \tag{3.12}$$

so that the same results as in Theorem 1 are achieved. □

Remark 7: In Theorem 3 the restrictive hypothesis H1 is adopted for the sake of simplicity. Actually what is necessary is just that the system equation (3.6) for the observer admits a unique solution such that

$$\left.\frac{\mathrm{d}\phi(x, D)}{\mathrm{d}x}\right|_{(\hat{x}(t),\hat{D}(t))}$$

has full rank. □

Also in this case the hypotheses H1 and H2 can be considerably weakened when the evolution of the system state $x(t)$ is confined in a bounded closed sphere $S(\rho_x)$, around the origin in $\mathbb{R}^n$ for any input $u \in U_a$ with

$$U_a = \left\{u \in U^{(r)}: \sum_{i=0}^{r} |u^{(i)}(t)|^2 \leqslant \rho_u^2\right\}$$

We can state the following result.

Corollary 3: *The estimation error for the observer* (3.6) *converges exponentially to zero if*:

(i) $\psi_{n+1}(\Phi^{-1}(\zeta, D), D)$ *is locally Lipschitz with respect to* $(\zeta, D)^{\mathrm{T}}$

(ii) $x(t) \in S(\rho_x)$, *for all* $t \geqslant 0$

(iii) $u(t) \in U_a$, *for all* $t \geqslant 0$

(iv) $Q(x, D)$ *has full rank for all* $(x, D)^{\mathrm{T}} \in S$ (radius $\{\Phi^{-1}(S(Nr + r))\}$) *for an arbitrarily fixed* $N > 1$ *and* r *is such that* $\|\Phi(x, D\| \leqslant r$, *for all* $x \in S(\rho_x)$, *for all* $u \in U_a$

(v) $\|\hat{x}(0) - x(0)\| \leqslant \beta$, *for a suitable* $\beta > 0$.

Proof: The proof, with slight modifications, proceeds along the same lines as Corollary 1, with references to the system equations (3.10). □

4. Examples of applications and concluding remarks

(*a*) *Predator–prey Volterra model with predator measurements.* This example satisfies the condition of Corollary 1, as the existence has been proved of a bounded invariant set for the evolution of the state which depends only on the initial conditions.

The system equations are as follows:

$$\dot{x}_1 = -ax_1 + bx_1x_2$$

$$\dot{x}_2 = cx_2 - dx_1x_2$$

$$y = x_1$$

The observer equations are as follows:

$$\dot{\hat{x}}_1 = -a\hat{x}_1 + b\hat{x}_1\hat{x}_2 + k_1(y - \hat{x}_1)$$

$$\dot{\hat{x}}_2 = c\hat{x}_2 - d\hat{x}_1\hat{x}_2 + \frac{(a - b\hat{x}_2)k_1 + k_2}{b\hat{x}_1}(y - \hat{x}_1)$$

In Fig. 1, an application example is reported in which $a = b = c = d = 1$.

(*b*) Here we show an application example that is readily recognized to satisfy the corollary of Theorem 2. A simple one-link robot arm is considered whose

rotary motion about one end is controlled by means of an elastically coupled actuator (see Isidori 1989, pp. 230–231).

The system is described by means of two second-order differential equations, one characterizing the mechanical balance of the actuator shaft, and the other characterizing the mechanical balance of the link:

$$J_1\ddot{q}_1 + F_1\dot{q}_1 + \frac{H}{N}\left(q_2 - \frac{q_1}{N}\right) = T$$

$$J_2\ddot{q}_2 + F_2\dot{q}_2 + H\left(q_2 - \frac{q_1}{N}\right) + mgd\cos q_2 = 0$$

where q_i denote the angular positions with respect to a fixed reference frame of, respectively, the actuator shaft and the link; J_i represent inertia; F_i are viscous friction constants; H is the elasticity constant of the spring; N is the transmission gear ratio; T is the torque produced at the actuator axis; m is the mass; and d is the position of the centre of gravity of the link. Then, choosing $x = [q_1 q_2 \ \dot{q}_1 \dot{q}_2]^{\mathrm{T}}$, we obtain the system

$$\dot{x}(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ -\dfrac{H}{J_1N^2}x_1(t) + \dfrac{H}{J_1N}x_2(t) - \dfrac{F_1}{J_1}x_3(t) \\ \dfrac{H}{J_2N}x_1(t) - \dfrac{H}{J_2}x_2(t) - \dfrac{mgd}{J_2}\cos x_2(t) - \dfrac{F_2}{J_2}x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_1} \\ 0 \end{bmatrix} u(t)$$

$$y(x(t)) = h(x(t)) = x_2(t)$$

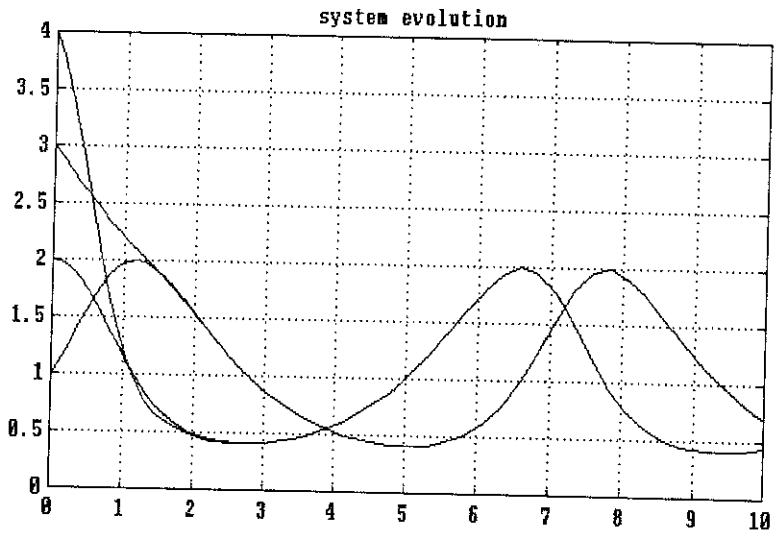
The observer equations are as follows:

$$\dot{\hat{x}}_1 = \hat{x}_3 + \left\{\left[N - \frac{Nmgd}{H}\sin\hat{x}_2\right]k_1 + \frac{NF_2}{H}k_2 + \frac{NJ_2}{H}k_3\right\}(y - \hat{x}_2)$$

$$\dot{\hat{x}}_2 = \hat{x}_4 + k_1(y - \hat{x}_2)$$

$$\begin{aligned}\dot{\hat{x}}_3 = &-\frac{H}{J_1N^2}\hat{x}_1 + \frac{H}{J_1N}\hat{x}_2 - \frac{F_1}{J_1}\hat{x}_3 + \frac{1}{J_1}u \\ &+ \left\{-\frac{N}{H}(\hat{x}_4\, mgd\cos\hat{x}_2)k_1\right. \\ &\left. + \left[N - \frac{Ndgm\sin\hat{x}_2}{H}\right]k_2 + \frac{NF_2}{H}k_3 + \frac{NJ_2}{H}k_4\right\}(y - \hat{x}_2)\end{aligned}$$

$$\dot{\hat{x}}_4 = \frac{H}{J_2N}\hat{x}_1 - \frac{H}{J_2}\hat{x}_2 - \frac{mgd}{J_2}\cos\hat{x}_2 - \frac{F_2}{J_2}\hat{x}_4 + k_2(y - \hat{x}_2)$$



$\lambda_1 = -2$ $\lambda_2 = -3$

Figure 1. Predator–Prey Volterra's model with predator measurements.

In Fig. 2 an application example in which, for the sake of simplicity, all the constants are set equal to one is reported. The input is $u(t) = 2 + \sin(t)$.

(*c*) The following example satisfies the conditions of Corollary 3. Given a d.c. motor (Isidori 1989, pp. 226–230) with constant rotor voltage and stator voltage as control variable, the system obtained has the equations

$$L_s \frac{dI_s}{dt} + R_s I_s = V_s$$

$$L_r \frac{dI_r}{dt} + R_r I_r = V_r - E$$

$$J \frac{d\Omega}{dt} + F\Omega = T$$

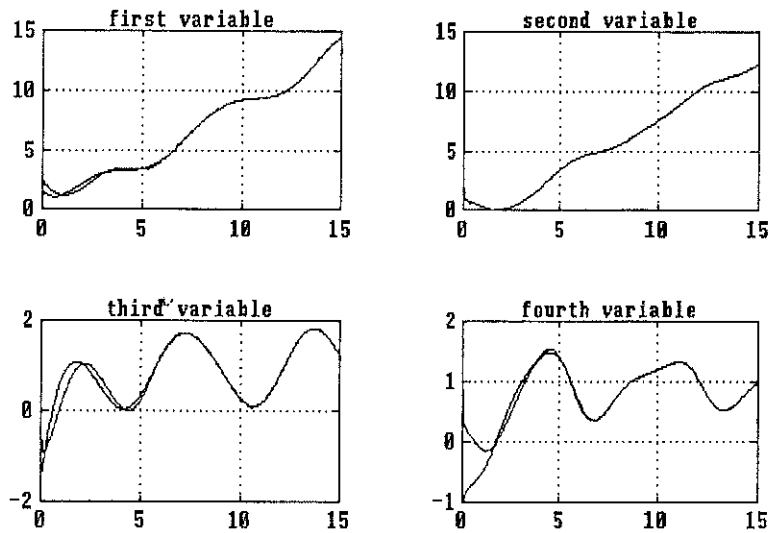
where I_s is the stator current; V_s is the stator voltage; R_s is the stator resistance; L_s is the stator winding inductance; I_r is the rotor current; V_r is the rotor voltage (constant by assumption); R_r is the rotor winding resistance; L_r is the rotor winding inductance; E is the back e.m.f.; Ω is the angular velocity of the motor shaft; J is the inertia of the load; F is the viscous friction constant; T is the torque developed at the shaft; and Φ is the flux associated with stator winding.

The coupling between the three equations is expressed by the relations

$$E = K_e \Phi \Omega$$

$$T = K_m \Phi I_r$$

$$\Phi = L_s I_s$$



$\lambda_1 = -2 \quad \lambda_2 = -2.1 \quad \lambda_3 = -2.2 \quad \lambda_4 = -2.3$

Figure 2. One link robot arm.

In the ideal hypothesis of 100% energy conversion efficiency:

$$K_e = K_m = K$$

Choosing the state variables

$$x_1 = I_s \quad x_2 = I_r \quad x_3 = \Omega$$

and $u = V_s$ we obtain the functions

$$f(x) = \begin{bmatrix} -\dfrac{R_s}{L_s} x_1 \\ -\dfrac{R_r}{L_r} x_2 + \dfrac{V_r}{L_r} - K \dfrac{L_s}{L_r} x_1 x_3 \\ -\dfrac{F}{J} x_3 + K \dfrac{L_s}{J} x_1 x_2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} \dfrac{1}{L_s} \\ 0 \\ 0 \end{bmatrix}$$

$$h(x) = \Omega - \Omega^0 = x_3 - \Omega^0$$

where Ω^0 is a fixed reference value. We have chosen the input to belong to the

class of piecewise constant functions; in this case,

$$\Phi(x, D) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

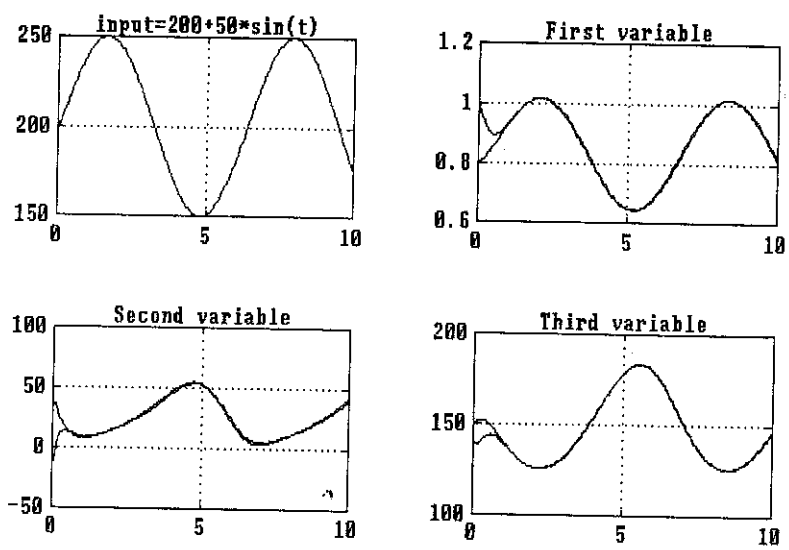
$$= \begin{bmatrix} x_3 - \Omega^0 \\ -\dfrac{F}{J} x_3 + \dfrac{L_s K}{J} x_1 x_2 \\ -\left(\dfrac{R_s}{L_s} + \dfrac{F}{J} + \dfrac{R_r}{L_r}\right) \dfrac{L_s K}{J} x_1 x_2 + \dfrac{K}{J} D x_2 + \dfrac{V_r}{L_r} \dfrac{L_s K}{J} x_1 - \dfrac{L_s^2 K^2}{J L_r} x_1^2 x_3 + \dfrac{F^2}{J^2} x_3 \end{bmatrix}$$

In Figs. 3 and 4, application examples are reported for different input signals. In these examples

$$\frac{R_s}{L_s} = 2; \quad \frac{R_r}{L_r} = 50; \quad \frac{V_r}{L_r} = 20000; \quad \frac{KL_s}{L_r} = 150; \quad KL_s = 1{\cdot}8$$

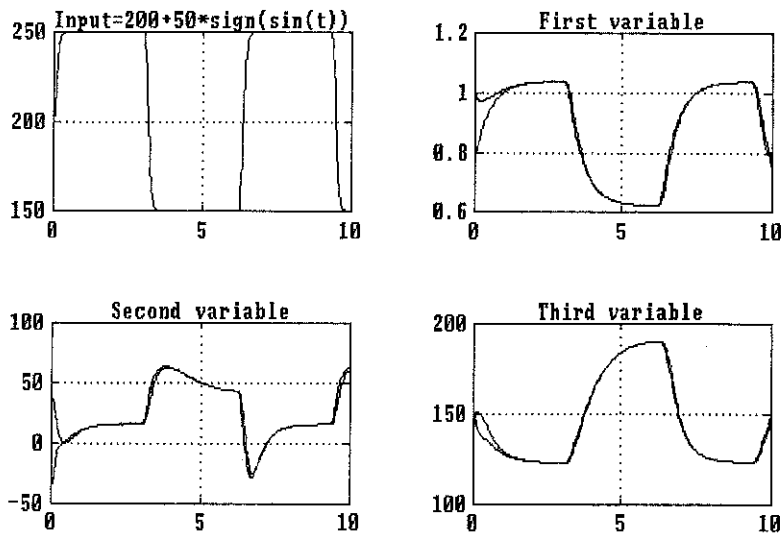
$$L_s = 120; \quad \frac{F}{J} = 0{\cdot}25; \quad J = 1$$

As concluding remarks we would like to stress that the proposed observer is easy to implement and that gives good results in practical applications. A possible improvement of this observer could be made by a suitable choice of the eigenvalues in order to minimize a given quality functional, which for example could take into account the precision of the output measurements.



$$\mathbf{k}_1 = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \qquad \mathbf{k}_2 = 15$$

Figure 3. DC motor (case 1).



$$\mathbf{k}_1 = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \qquad \mathbf{k}_2 = 15$$

Figure 4. DC motor (case 2).

Acknowledgments

We should like to thank Professor A. Isidori for introducing us to non linear system theory through his fundamental book on nonlinear control systems and for his valuable suggestions and fruitful discussions. We also thank Professor S. Monaco for his advice.

This work was supported to the extent of 60% by the Ministero Université Ricerce Scientifice & Tecnologice (MURST).

References

Bastin, G., and Gevers, M. R., 1988, Stable adaptive observers for nonlinear time-varying systems. *IEEE Transactions on Automatic Control*, **33**, 650–658.

Brunowsky, P., 1970, A classification of linear controllable systems. *Kybernetika*, **6**, 173–188.

Gauthier, J. P., Hammouri, H. and Othman, S., 1992, A simple observer for nonlinear systems — applications to a bioreactors. *IEEE Transactions on Automatic Control*, **37**, 875–880.

Isidori, A., 1989, *Nonlinear Control Systems: an Introduction* (Berlin: Springer-Verlag).

Krener, A. J., and Isidori, A., 1983, Linearization by output injection and nonlinear observers. *Systems and Control Letters*, **3**, 47–52.

Krener, A. J., and Respondek, W., 1985, Nonlinear observers with linearizable error dynamics. *SIAM Journal on Control and Optimization* **23**, 197–216.

Lakshmikantham, V., Leela, S., and Artynuk, A. A. M., 1989, *Stability Analysis of Nonlinear Systems* (New York: Marcel Dekker).

Luenberger, D. G., 1971, An Introduction to observers. *IEEE Transactions on Automatic Control*, **16**, 592–602.

Marino, R., 1990, Adaptive observer for single output nonlinear systems. *IEEE Transactions on Automatic Control*, **35**, 1054–1058.