MRE numerical algorithm and its steady-state solution under particular conditions. A full explanation of this trend is difficult, because of the complexity of the data association process built in the MSPDA algorithm, but one explanation would be that processing the best sensor last improves tracking performance of the sensor system as a whole. Analyses over ranges of sensor parameters show that tracking system performance of the sequential MSJPDA filter, in terms of the rms position error, favors using sensors of comparable qualities, and that processing the worse sensor first gives better results if the sensor qualities do not differ by a large amount.

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Design of State Observers from a Drift-Observability Property

Manuela Dalla Mora, Alfredo Germani, and Costanzo Manes

***Abstract*—In this paper, the problem of state observation with exponential error decay for nonlinear systems affine in the input is considered and an observer is proposed. For such an observer, the drift-observability property of the system (*i.e.*, observability for zero input), together with an assumption on the input amplitude or on the observation relative degree, are sufficient conditions for exponential convergence of the observation error. The existence of an exponential observer is correlated to the existence of a solution for an H_∞ Riccati-like inequality. Global and semiglobal convergence results are presented.**

***Index Terms*—Nonlinear observers, nonlinear systems, observability.**

I. INTRODUCTION

Most recent contributions in the field of state observation for nonlinear systems can be roughly classified into two categories. In one, the geometrical properties of the vector fields defining the system are exploited and the powerful tools of differential geometry are extensively used [1], [2], [7], [10], [14]–[16], [19], [20], [23]. In the other category of works, the state observation problem is studied considering functional parameters of nonlinear systems, such as norms, gains and Lipschitz constants [9], [17], [21], [25]–[28]. Most works of both categories provide a solution of the state observation problem only for *classes* of systems.

Among the works of the first category, the papers [1], [13], [16], [19], and [20] consider the class of nonlinear systems in which the nonlinearity is given by an output injection (conditionally linear systems). For such systems, a complete observation theory is available, which also includes adaptive observers. In [8], [10], a characterization of the class of systems observable for any input is given, and for these, an exponential observer is provided. Stronger hypotheses are required for the construction of observers for the class of systems considered in [14], [15] that admit linear error dynamics. In [23], [24], a high-gain observer of the type presented in [6] is used inside a control loop. This kind of observer, like the one introduced in [26], requires the explicit computation of the state as a function of the input and output derivatives. When the control loop is closed, the input derivatives are available while the output derivatives must be estimated. The paper [23] also considers observers for nonlinear systems that have a particular block-triangular structure, also including the class of uniformly observable systems [8], [10]. In [2], the observation problem is solved for nonlinear systems with full relative degree. Reference [9] considers observers for the class of bilinear systems, with linear dissipative drift.

Most of the works of the second category consider linear observable systems with nonlinear perturbation terms characterized by a Lipschitz constant and/or by a finite bound on the norm. In [21], [27], nonlinear systems in which the output is a linear function of the state are considered. In [27], for such a class, the existence of an exponential observer is implied by the existence of a solution for a Lyapunov-like inequality involving the Jacobian of the drift term. In [21], a linear part can be extracted from the drift and the existence of the observer is implied by the existence of a solution for an algebraic Riccati equation. In [28], the result of [27] is extended to a larger class of output functions. In [17], an observer is presented for the class of systems with linear and stable drift, and nonlinearity only in the forcing term.

All of the above-mentioned papers restrict the analysis to classes of nonlinear systems, characterized by geometrical properties or by the particular structure of the dynamic equations. The aim of this paper is to consider general nonlinear systems affine in the control and to fully exploit the property of *drift-observability* for the construction of an observer that works for meaningful classes of input functions. The nonlinear systems considered here have the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1.1a}$$

$$y(t) = h(x(t)) \tag{1.1b}$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$. $g(x) = [g_1(x), \cdots, g_p(x)]$ is a matrix whose columns are $C^k(X)$ vector fields, $f(x)$ is a $C^k(X)$ vector field, and the vector function $h(x)$ is $C^k(X)$ too, where k is an integer that allows all differentiations needed in the paper.

II. PRELIMINARIES

Definition 2.1: A function $\lambda(x)$ is said to be *affine linear growth (ALG)* in a set S if positive constants γ and b exist such that $\|\lambda(x_1) - \lambda(x_2)\|^2 \leq \gamma^2\|x_1 - x_2\|^2 + b^2$, $\forall x_1, x_2 \in S$.

Remark 2.2: It is clear that a function uniformly Lipschitz in S is ALG in the same set, with $b = 0$. It can be readily proven also that functions uniformly Hölder in S are ALG in the same set. Recall that

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The authors are with the Dipartimento di Ingegneria Elettrica, Università degli studi dell'Aquila, Monteluco di Roio, L'Aquila, Italy (e-mail: germani@ing.univaq.it; manes@ing.univaq.it).

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$\lambda(x)$ is uniformly Hölder in S if a positive h and a constant $\delta \in (0,1)$ exist such that $\|\lambda(x_1) - \lambda(x_2)\| \le h\|x_1 - x_2\|^\delta, \forall x_1, x_2 \in S$. ■

Recall that the Lie derivative of order k (k nonnegative integer) of a scalar function $\lambda(x)$ along a vector field $f(x)$ is recursively defined as follows:

$$L_f^0\lambda(x) \triangleq \lambda(x),$$
$$L_f^k\lambda(x) \triangleq \frac{\partial L_f^{k-1}\lambda}{\partial x} f(x), \qquad k = 1, 2, \cdots \tag{2.1}$$

For each $j = 1, \cdots, q$, consider the vector function

$$\Phi_j^{s_j}(x) \triangleq [h_j(x) \quad L_f h_j(x) \quad \cdots \quad L_f^{s_j-1} h_j(x)]^{\mathrm{T}}. \tag{2.2}$$

Let $\overline{s} = (s_1, \cdots, s_q)$ be a multi-index such that $\sum_{j=1}^q s_j = n$. Consider the following square map

$$\Phi_{\overline{s}}(x) \triangleq [\Phi_1^{s_1}(x)^{\mathrm{T}} \quad \cdots \quad \Phi_q^{s_q}(x)^{\mathrm{T}}]^{\mathrm{T}}. \tag{2.3}$$

Denoting with $Y_{\overline{s}}$ the vector of output derivatives

$$Y_{\overline{s}} = [y_1 \quad \cdots \quad y_1^{(s_1-1)} \quad \cdots \quad y_q \quad \cdots \quad y_q^{(s_q-1)}]^{\mathrm{T}} \tag{2.4}$$

if $u(t) \equiv 0$, it is $Y_{\overline{s}}(t) = \Phi_{\overline{s}}(x(t))$, and, therefore, invertibility of the map $\Phi_{\overline{s}}$ and the knowledge of vector $Y_{\overline{s}}$ allow exact state reconstruction. This property justifies the following definition.

Definition 2.3: A map $\Phi_{\overline{s}}(x)$ is said to be an *observability map* in a set $\Omega \subseteq \mathbb{R}^n$ if it is a diffeomorphism in an open set that contains or coincides with Ω. A system that admits an observability map in Ω for a given $\overline{s}$ is said to be *drift-observable* in Ω. ■

It follows that the Jacobian associated to an observability map

$$Q_{\overline{s}}(x) \triangleq \frac{\partial \Phi_{\overline{s}}(x)}{\partial x} \tag{2.5}$$

is nonsingular in Ω, and the inverse map of $z = \Phi_{\overline{s}}(x)$ exists in $\Phi_{\overline{s}}(\Omega)$ and is denoted as $x = \Phi_{\overline{s}}^{-1}(z)$. Although, in general, such a map is difficult to compute, its Jacobian can be easily computed as

$$\left.\frac{\partial \Phi_{\overline{s}}^{-1}(z)}{\partial z}\right|_{z=\Phi_{\overline{s}}(x)} = Q_{\overline{s}}^{-1}(x). \tag{2.6}$$

Definition 2.4: A system is said to be *uniformly Lipschitz drift-observable* (ULDO) in a set $\Omega \subseteq \mathbb{R}^n$ if it is drift-observable in Ω and the maps $\Phi_{\overline{s}}$ and $\Phi_{\overline{s}}^{-1}$ are uniformly Lipschitz in Ω and $\Phi_{\overline{s}}(\Omega)$, respectively. If $\Omega \equiv \mathbb{R}^n$, the system is said to be *globally uniformly Lipschitz drift-observable* (GULDO).

In the state observation theory, it is useful to note the *observation relative degree* (a concept weaker then *relative degree*). The following notation is needed

$$L_g L_f^s h_j(x) \triangleq [L_{g_1} L_f^s h_j(x) \quad \cdots \quad L_{g_p} L_f^s h_j(x)]. \tag{2.7}$$

Definition 2.5: The *observation relative degree* r_j of the jth output of system (1.1) in a set $\Omega \subseteq \mathbb{R}^n$ is an integer such that

$$\forall x \in \Omega: \quad L_g L_f^s h_j(x) = 0, \qquad s = 0, 1, \cdots, r_j - 2$$
$$\exists x \in \Omega: \quad L_g L_f^{r_j-1} h_j(x) \neq 0. \tag{2.8}$$

The multi-index $\overline{r} = (r_1, \cdots, r_q)$ is called *vector observation relative degree*. ■

The product of $Q_{\overline{s}}(x)$ with the matrix $g(x)$ has an interesting structure. Consider first the product

$$\frac{\partial \Phi_j^{s_j}(x)}{\partial x} g(x) = \left[L_g h_j(x), \quad \cdots, \quad L_g L_f^{s_j-1} h_j(x)\right]^{\mathrm{T}}. \tag{2.9}$$

Note that the first $r_j - 1$ rows of matrix (2.9) are identically zero in Ω (if $s_j < r_j$, the product (2.9) is zero). Without loss of generality, assume the observability map such that $s_j \ge r_j$ for the first $\overline{q}$ outputs and $s_j < r_j$ for the remaining $q - \overline{q}$, so that the last $\sum_{j=\overline{q}+1}^q s_j$ row blocks of the product $Q_{\overline{s}}(x)g(x)$ are identically zero. On the contrary, for the first $\overline{q}$ row blocks, one has the following structure:

$$\frac{\partial \Phi_j^{s_j}(x)}{\partial x} g(x) = \begin{bmatrix} 0_{(r_j-1)\times p} \\ H_j(x) \end{bmatrix}, \qquad j = 1, \cdots, \overline{q}. \tag{2.10}$$

The matrix $H_j(x)$ is $d_j \times p$, with $d_j \triangleq s_j - r_j + 1$, and is defined as

$$H_j(x) \triangleq \left[L_g L_f^{r_j-1} h_j(x), \quad \cdots, \quad L_g L_f^{s_j-1} h_j(x)\right]^{\mathrm{T}}. \tag{2.11}$$

Defining the multi-index $\overline{d} \triangleq (d_1, \cdots, d_{\overline{q}})$, and the integer $d \triangleq \sum_{j=1}^{\overline{q}} d_j$, a $n \times d$ matrix $F_{\overline{s},\overline{r}}$ of zeros and ones can be defined as

$$F_{\overline{s},\overline{r}} = \begin{bmatrix} \mathrm{diag}_{j=1}^{\overline{q}}\{S_j\} \\ 0_{\tilde{s}\times d} \end{bmatrix} \quad \text{where} \quad S_j = \begin{bmatrix} 0_{(r_j-1)\times d_j} \\ I_{d_j} \end{bmatrix} \tag{2.12}$$

where I_{d_j} denotes the $d_j \times d_j$ identity matrix and $\tilde{s} = \sum_{j=\overline{q}+1}^q s_j$.

Now, the product $Q_{\overline{s}}(x)g(x)$ can be definitively written as

$$Q_{\overline{s}}(x)g(x) = F_{\overline{s},\overline{r}} H_{\overline{s},\overline{r}}(x), \quad \text{where}$$
$$H_{\overline{s},\overline{r}}(x) \triangleq \left[H_1^{\mathrm{T}}(x), \quad \cdots, \quad H_{\overline{q}}^{\mathrm{T}}(x)\right]. \tag{2.13}$$

Now, let $L_{\overline{s}}(x) \triangleq [L_f^{s_1} h_1(x), \cdots, L_f^{s_q} h_q(x)]^{\mathrm{T}}$. It is useful to define the following vector and matrix functions, defined in $\Phi_{\overline{s}}(\Omega)$:

$$\overline{L}_{\overline{s}}(z) \triangleq L_{\overline{s}}(\Phi_{\overline{s}}^{-1}(z)), \quad \overline{H}_{\overline{s},\overline{r}}(z) \triangleq H_{\overline{s},\overline{r}}(\Phi_{\overline{s}}^{-1}(z)). \tag{2.14}$$

It is easy to check through direct computation that a drift-observable system (1.1) can be rewritten using the coordinate transformation $z = \Phi_{\overline{s}}(x)$ as

$$\dot{z} = A_{\overline{s}} z + B_{\overline{s}} \overline{L}_{\overline{s}}(z) + F_{\overline{s},\overline{r}} \overline{H}_{\overline{s},\overline{r}}(z) u$$
$$y = C_{\overline{s}} z \tag{2.15}$$

where $A_{\overline{s}}$, $B_{\overline{s}}$, and $C_{\overline{s}}$ are block-diagonal matrices $A_{\overline{s}} = \mathrm{diag}_{j=1}^q\{A_j\}$, $B_{\overline{s}} = \mathrm{diag}_{j=1}^q\{B_j\}$, $C_{\overline{s}} = \mathrm{diag}_{j=1}^q\{C_j\}$, in which the q triples (A_j, B_j, C_j) are Brunowsky matrices

$$A_j = \begin{bmatrix} 0_{(s_j-1)\times 1} & I_{s_j-1} \\ 0 & 0_{1\times(s_j-1)} \end{bmatrix}, \quad B_j = \begin{bmatrix} 0_{(s_j-1)\times 1} \\ 1 \end{bmatrix},$$
$$C_j = [1 \quad 0_{1\times(s_j-1)}]. \tag{2.16}$$

As it will be shown in Section III, the existence of an exponential observer for system (1.1) is related with the existence of a solution for the following H_∞ Riccati-like matrix inequality:

$$(A_{\overline{s}} - KC_{\overline{s}})P + P(A_{\overline{s}} - KC_{\overline{s}})^{\mathrm{T}} + B_{\overline{s}} B_{\overline{s}}^{\mathrm{T}}$$
$$+ \rho^2 F_{\overline{s},\overline{r}} F_{\overline{s},\overline{r}}^{\mathrm{T}} + 2\alpha P + \gamma^2 P^2 \le 0 \tag{2.17}$$

where ρ, α, and γ are given positive real parts and $K \in \mathbb{R}^{n\times q}$ and $P \in \mathbb{R}^{n\times n}$ are the unknown matrices.

Definition 2.6: A pair (K, P) is said to be a *stabilizing solution pair* if it satisfies inequality (2.17), all eigenvalues of $A_{\overline{s}} - KC_{\overline{s}}$ have negative real parts and P is symmetric positive definite. ■

The following theorems state some relationships between existence of a stabilizing solution pair (K, P) and the values assumed by the triple (ρ, α, γ) of coefficients. The proofs are extensions of those in [4] for the single-input/single-output (SISO) case. For further details, see [3].

Theorem 2.7: For any $\alpha \ge 0$ and $\gamma \ge 0$, a strictly positive $\overline{\rho}$ exists such that for any $\rho \le \overline{\rho}$, the H_∞ Riccati-like matrix inequality (2.17) admits a stabilizing solution pair (K, P).

Theorem 2.8: For any $\alpha \ge 0$ and $\rho \ge 0$, a positive $\overline{\gamma}$ exists such that for any $\gamma \le \overline{\gamma}$, the H_∞ Riccati-like matrix inequality (2.17) admits a stabilizing solution pair (K, P).

Theorem 2.9: If $\overline{s} \le \overline{r}$ (componentwise), then for any positive triple (ρ, α, γ), the H_∞ Riccati-like matrix inequality (2.17) admits a stabilizing solution pair (K, P).

Theorem 2.10: Consider the H_∞ Riccati-like matrix inequality (2.17) with given positive ρ and γ. If $\overline{s} \le \overline{r}$, then for any $\varepsilon > 0$ a positive α to be put in (2.17) and a stabilizing solution pair (K, P) exist such that $\|P\|/\alpha \le \varepsilon$.

Remark 2.11: In the proofs of all the theorems above, the pair (K, P) that solves (2.17) is provided (see [3], [4]). The structure of K

is chosen block-diagonal, so that $A_{\overline{s}} - KC_{\overline{s}}$ is in block-companion form

$$K = \mathrm{diag}_{j=1}^{q}\{K_j\}, \qquad K_j = [k_{j,1} \quad \cdots \quad k_{j,s_j}]^{\mathrm{T}}. \tag{2.18}$$

Let $K_j(\overline{\lambda}_j)$ denote the vector gain that assigns eigenvalues $\overline{\lambda}_j = (\lambda_{j,1}, \cdots, \lambda_{j,s_j})$ to the jth block $A_j - K_jC_j$. If the eigenvalues of each block are distinct, this is diagonalized by a Vandermonde matrix, denoted as $V(\overline{\lambda}_j)$, and, therefore, matrix $A_{\overline{s}} - KC_{\overline{s}}$ is diagonalized by the block Vandermonde matrix $V_{\overline{s}}(\overline{\lambda}) \triangleq \mathrm{diag}_{j=1}^{q}\{V(\overline{\lambda}_j)\}$, where $\overline{\lambda} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_q)$ is the set of eigenvalues assigned to $A_{\overline{s}} - KC_{\overline{s}}$. Given a positive parameter w, consider the set of eigenvalues $\overline{\lambda}(w) \triangleq (\overline{\lambda}_1(w), \cdots, \overline{\lambda}_q(w))$, where $\overline{\lambda}_j(w) = (-w, \cdots, -w^{s_j})$. Let $K(w)$ denote the gain matrix, with structure (2.18), that assigns the set $\overline{\lambda}(w)$ of eigenvalues to matrix $A_{\overline{s}} - K(w)C_{\overline{s}}$, and let $V_{\overline{s}}(w)$ denote the block-Vandermonde matrix that diagonalizes matrix $A_{\overline{s}} - K(w)C_{\overline{s}}$. In all proofs of Theorems 2.7, 2.8, 2.9, and 2.10, it is shown that the pair $(K(w), P(w))$, with $P(w) = (V_{\overline{s}}^{\mathrm{T}}(w)V_{\overline{s}}(w))^{-1}$, solves the H_∞ Riccati-like inequality of the theorems for a sufficiently large value of the parameter w. ■

III. Exponential Observer for Nonlinear Systems

In this section, it is proven that a dynamic system of the type

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + Q_{\overline{s}}^{-1}(\hat{x}(t))K(y(t) - h(\hat{x}(t))) \tag{3.1}$$

is an exponential observer for system (1.1), provided that some conditions both on the system and on the input u are satisfied and that the gain matrix K is properly chosen. The basic assumption for the existence of an exponential observer is the drift-observability of the system. The rather technical assumption of Uniform Lipschitz Drift-Observability (ULDO) is needed too, together with the following:

Assumption $\mathcal{H}^0$: (ALG nonlinearities). The matrix functions $\overline{L}_{\overline{s}}(z)$ and $\overline{H}_{\overline{s},\overline{r}}(z)$ defined in (2.14) are ALG in a set S (see Definition 2.1). Let $\gamma_{\overline{L}}$, $\gamma_{\overline{H}}$ and $b_{\overline{L}}$, $b_{\overline{H}}$ be the relevant constants.

All theorems in the following will be stated under the assumption $\mathcal{H}^0$; that is quite general, but it does not give strong convergence results for the observer. Stronger results on convergence are obtained if the following assumption is made instead:

Assumption $\mathcal{H}^1$: (Lipschitz nonlinearities). The matrix functions $\overline{L}_{\overline{s}}(z)$ and $\overline{H}_{\overline{s},\overline{r}}(z)$ defined in (2.14) are uniformly Lipschitz in a set S. Let $\gamma_{\overline{L}}$ and $\gamma_{\overline{H}}$ be the Lipschitz constants.

Being that $\mathcal{H}^1$ is a particular case of $\mathcal{H}^0$, in the following, all theorems will be stated only under $\mathcal{H}^0$. Global and semiglobal results on state observation are presented.

A. Global Observer

Theorem 3.1: Assume that system (1.1) is GULDO and that the property $\mathcal{H}^0$ holds in $\mathbb{R}^n$. Then, for any $\alpha > 0$, a gain matrix $K \in \mathbb{R}^{n\times q}$ for the observer (3.1) and strictly positive constants μ_1, μ_2 and u_M exist such that if $\|u(t)\| \le u_M$, $\forall t \ge 0$, then

$$\|x(t) - \hat{x}(t)\| \le \mu_1 e^{-\alpha t}\|x(0) - \hat{x}(0)\| + \mu_2\sqrt{b_{\overline{L}}^2 + b_{\overline{H}}^2}, \quad \forall x(0), \hat{x}(0) \in \mathbb{R}^n. \tag{3.2}$$

Proof: Consider the coordinate transformation given by the observability map, $z = \Phi_{\overline{s}}(x)$ and $\hat{z} = \Phi_{\overline{s}}(\hat{x})$. For the observation errors $e_x \triangleq x - \hat{x}$ and $e_z \triangleq z - \hat{z}$, in all $\mathbb{R}^n$, it is

$$\|e_z\| \le \gamma_{\Phi_{\overline{s}}}\|e_x\|, \quad \|e_x\| \le \gamma_{\Phi_{\overline{s}}^{-1}}\|e_z\|. \tag{3.3}$$

In z-coordinates, system (1.1) becomes (2.15), and the observer (3.1) becomes

$$\dot{\hat{z}} = A_{\overline{s}}\hat{z} + B_{\overline{s}}\overline{L}_{\overline{s}}(\hat{z}) + F_{\overline{s},\overline{r}}\overline{H}_{\overline{s},\overline{r}}(\hat{z})u + K(y - C_{\overline{s}}\hat{z}). \tag{3.4}$$

The dynamics of the observation error in z-coordinates is

$$\dot{e}_z = (A_{\overline{s}} - KC_{\overline{s}})e_z + B_{\overline{s}}v_1(z,\hat{z}) + F_{\overline{s},\overline{r}}v_2(z,\hat{z})u, \tag{3.5}$$

with

$$\begin{aligned} v_1(z,\hat{z}) &\triangleq \overline{L}_{\overline{s}}(z) - \overline{L}_{\overline{s}}(\hat{z}), \\ v_2(z,\hat{z}) &\triangleq \overline{H}_{\overline{s},\overline{r}}(z) - \overline{H}_{\overline{s},\overline{r}}(\hat{z}). \end{aligned} \tag{3.6}$$

Being ALG, the functions $\overline{L}_{\overline{s}}(z)$ and $\overline{H}_{\overline{s},\overline{r}}(z)$ the perturbations are such that $\forall z, \hat{z} \in \mathbb{R}^n$

$$\begin{aligned} \|v_1(z,\hat{z})\|^2 &\le \gamma_{\overline{L}}^2\|e_z\|^2 + b_{\overline{L}}^2, \\ \|v_2(z,\hat{z})\|^2 &\le \gamma_{\overline{H}}^2\|e_z\|^2 + b_{\overline{H}}^2. \end{aligned} \tag{3.7}$$

Consider now the H_∞ Riccati-like inequality (2.17) with prescribed α and $\gamma^2 = \gamma_{\overline{L}}^2 + \gamma_{\overline{H}}^2$. From Theorem 2.7, a positive $\overline{\rho}$ exists such that the inequality admits a stabilizing solution pair (K, P). Take such $\overline{\rho}$ as the bound on the input function, *i.e.*, set $u_M \triangleq \overline{\rho}$, and take K as the observer gain in (3.1). Use the symmetric positive definite matrix P to define the following positive definite function of the error e_z

$$\nu(e_z) = e_z^{\mathrm{T}}P^{-1}e_z. \tag{3.8}$$

The derivative of ν along the error trajectory is

$$\begin{aligned} \dot{\nu}(z,\hat{z}) = {} & e_z^{\mathrm{T}}(P^{-1}(A_{\overline{s}} - KC_{\overline{s}}) + (A_{\overline{s}} - KC_{\overline{s}})^{\mathrm{T}}P^{-1})e_z \\ & + 2e_z^{\mathrm{T}}P^{-1}B_{\overline{s}}v_1(z,\hat{z}) + 2e_z^{\mathrm{T}}P^{-1}F_{\overline{s},\overline{r}}v_2(z,\hat{z})u. \end{aligned} \tag{3.9}$$

The following inequality that holds any pair of vectors a and b and for any real β

$$2a^{\mathrm{T}}b \le \beta^2\|a\|^2 + \frac{1}{\beta^2}\|b\|^2. \tag{3.10}$$

allows to write the inequality

$$\begin{aligned} \dot{\nu} \le {} & e_z^{\mathrm{T}}\Big(P^{-1}(A_{\overline{s}} - KC_{\overline{s}}) + (A_{\overline{s}} - KC_{\overline{s}})^{\mathrm{T}}P^{-1}\Big)e_z \\ & + e_z^{\mathrm{T}}P^{-1}B_{\overline{s}}B_{\overline{s}}^{\mathrm{T}}P^{-1}e_z + \|v_1\|^2 \\ & + u_M^2 e_z^{\mathrm{T}}P^{-1}F_{\overline{s},\overline{r}}F_{\overline{s},\overline{r}}^{\mathrm{T}}P^{-1}e_z + \frac{\|v_2\,u\|^2}{u_M^2}. \end{aligned} \tag{3.11}$$

Taking into account (3.7), and noting that $\|v_2 u\|^2/u_M^2 \le \|v_2\|^2$, it is

$$\begin{aligned} \dot{\nu}(z,\hat{z}) \le {} & e_z^{\mathrm{T}}P^{-1}\Big((A_{\overline{s}} - KC_{\overline{s}})P + P(A_{\overline{s}} - KC_{\overline{s}})^{\mathrm{T}} \\ & + B_{\overline{s}}B_{\overline{s}}^{\mathrm{T}} + u_M^2F_{\overline{s},\overline{r}}F_{\overline{s},\overline{r}}^{\mathrm{T}} + \gamma^2P^2\Big)P^{-1}e_z \\ & + b_{\overline{L}}^2 + b_{\overline{H}}^2. \end{aligned} \tag{3.12}$$

Exploiting inequality (2.17), satisfied by the current pair (K, P), it follows

$$\begin{aligned} \dot{\nu} &\le -2\alpha e_z^{\mathrm{T}}P^{-1}e_z + b_{\overline{L}}^2 + b_{\overline{H}}^2 \\ &= -2\alpha\nu + b_{\overline{L}}^2 + b_{\overline{H}}^2 \end{aligned} \tag{3.13}$$

$$\nu(e_z(t)) \le e^{-2\alpha t}\nu(e_z(0)) + \frac{1 - e^{-2\alpha t}}{2\alpha}\,(b_{\overline{L}}^2 + b_{\overline{H}}^2) \tag{3.14}$$

$$\begin{aligned} \|e_z(t)\|^2 \le {} & \frac{\lambda_{\max}(P^{-1})}{\lambda_{\min}(P^{-1})}e^{-2\alpha t}\|e_z(0)\|^2 \\ & + \frac{1 - e^{-2\alpha t}}{2\alpha}\cdot\frac{b_{\overline{L}}^2 + b_{\overline{H}}^2}{\lambda_{\min}(P^{-1})}. \end{aligned} \tag{3.15}$$

($\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the max and min eigenvalue of a matrix). From the Lipschitz properties (3.3), one obtains

$$\begin{aligned} \|e_x(t)\|^2 \le {} & (\gamma_{\Phi_{\overline{s}}}\gamma_{\Phi_{\overline{s}}^{-1}})^2\,\mathrm{cn}(P)e^{-2\alpha t}\|e_x(0)\|^2 \\ & + \frac{1 - e^{-2\alpha t}}{2\alpha}\,\|P\|\gamma_{\Phi_{\overline{s}}^{-1}}^2(b_{\overline{L}}^2 + b_{\overline{H}}^2) \end{aligned} \tag{3.16}$$

($\mathrm{cn}(P) = (\lambda_{\max}(P)/\lambda_{\min}(P)) = (\lambda_{\max}(P^{-1})/\lambda_{\min}(P^{-1}))$, $\|P\| = \lambda_{\max}(P) = (1/\lambda_{\min}(P^{-1}))$). This concludes the proof, since formula (3.2) is easily obtained with

$$\mu_1 = \gamma_{\Phi_{\overline{s}}}\gamma_{\Phi_{\overline{s}}^{-1}}\sqrt{\mathrm{cn}(P)}, \quad \mu_2 = \gamma_{\Phi_{\overline{s}}^{-1}}\sqrt{\|P\|/(2\alpha)}. \tag{3.17}$$

■

Remark 3.2: Theorem 3.1 states that for GULDO systems under assumption $\mathcal{H}^0$ (ALG nonlinearities), if the input is smaller than a suitable bound, an observer gain K can be found that ensures exponential decay of observation error at a desired rate α below a bound that depends on the constants $b_{\overline{L}}$ and $b_{\overline{H}}$. If the stronger assumption $\mathcal{H}^1$ (Lipschitz nonlinearities) is made in Theorem 3.1 ($b_{\overline{L}} = b_{\overline{H}} = 0$), the stronger result of exponential convergence of the observation error to zero is obtained. ■

Remark 3.3: Note that the conditions for existence of an exponential observer given in Theorem 3.1 *do not* include observability for any input but only *drift-observability*. It is the bound u_M imposed on the input that excludes the presence of inputs that make some system states indistinguishable. ■

As it can be seen in the proof of Theorem 3.1, the bound u_M on the input needed for the convergence of the observer depends on the constants $\gamma_{\overline{L}}$ and $\gamma_{\overline{H}}$ of the system, and the final bound on the observation error depends on the constants $b_{\overline{L}}$ and $b_{\overline{H}}$. The following theorem shows that if system (1.1) is such that the vector observation relative degree $\overline{r}$ is greater than the multi-index $\overline{s}$ that characterizes the map $\Phi_{\overline{s}}$, then for *any* given bound u_M on the input and for *any* desired decay rate α, an observer gain K can be found that guarantees convergence of the observation error below *any* desired asymptotic bound (convergence to zero in the case of assumption $\mathcal{H}^1$).

Theorem 3.4: Assume that system (1.1) is GULDO with vector observation relative degree $\overline{r} \geq \overline{s}$. Moreover, assume that property $\mathcal{H}^0$ holds in $\mathbb{R}^n$ -and that the input is bounded by a constant u_M ($\|u(t)\| \leq u_M$, $\forall t \geq 0$).

Then, for any positive $\overline{\alpha}$ and $\overline{b}$ an $\alpha \geq \overline{\alpha}$, a gain matrix $K \in \mathbb{R}^{n\times q}$ for the observer (3.1), and a strictly positive constant μ exist such that

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t}\|x(0) - \hat{x}(0)\| + \overline{b} \qquad \forall x(0), \hat{x}(0) \in \mathbb{R}^n. \tag{3.18}$$

Proof: Consider the H_∞ Riccati-like inequality (2.17) in which $\rho = u_M$ and $\gamma^2 = \gamma_{\overline{L}}^2 + \gamma_{\overline{H}}^2$. Recall that if $\overline{r} \geq \overline{s}$, then for any $\varepsilon > 0$, $\alpha > 0$ exists such that inequality (2.17) admits a stabilizing solution pair such that $\|P\|/\alpha \leq \varepsilon$ (Theorem 2.10). Therefore, an $\alpha \geq \overline{\alpha}$ and a solution pair (K, P) exist such that

$$\frac{\|P\|}{\alpha} \leq \frac{2\overline{b}^2}{\gamma_{\Phi_{\overline{s}}^{-1}}^2(b_{\overline{L}}^2 + b_{\overline{H}}^2)}. \tag{3.19}$$

The same steps made in the proof of Theorem 3.1 can be followed and inequality (3.2) can be obtained, in which μ_1 and μ_2 are given by (3.17). As a consequence, with the choice made for α and for the solution pair (K, P), it is

$$\mu_2\sqrt{b_{\overline{L}}^2 + b_{\overline{H}}^2} = \gamma_{\Phi_{\overline{s}}^{-1}}\sqrt{\frac{\|P\|}{2\alpha}} \cdot \sqrt{b_{\overline{L}}^2 + b_{\overline{H}}^2} \leq \overline{b} \tag{3.20}$$

and, therefore, setting $\mu = \mu_1$ in (3.2) inequality (3.18) is obtained and the theorem is proven. ■

Remark 3.5: As in the case of Theorem 3.1, if assumption $\mathcal{H}^1$ is made instead of $\mathcal{H}^0$, exponential convergence to zero of the observation error is obtained. ■

B. Semiglobal Observer

Semiglobal results on state observation can be obtained if for system (1.1) the ULDO property and hypotheses $\mathcal{H}^0$ (or $\mathcal{H}^1$) hold in subsets of $\mathbb{R}^n$, together with a bounded-input bounded-state (BIBS) stability property.

Definition 3.6: Let Ω be a set in $\mathbb{R}^n$. Given a positive δ, the set Ω_δ is defined as

$$\Omega_\delta \triangleq \{x \in \mathbb{R}^n \mid \|x - y\| \leq \delta, \forall y \in \Omega\}. \tag{3.21}$$

Definition 3.7: (BIBS stability) A system (1.1) is said to be *BIBS stable* if for any positive $\overline{u}$, two compact sets $\Omega^a \subseteq \Omega^b \subset \mathbb{R}^n$ exist such that if $x(0) \in \Omega^a$ and $\|u(t)\| \leq \overline{u}$, then $x(t) \in \Omega^b$, $\forall t \geq 0$. ■

An observer that ensures convergence to zero of the observation error when the state is confined in a bounded set Ω^b is called here a *semiglobal observer*. The semiglobal versions of Theorem 3.1 follows.

Theorem 3.8: Assume that system (1.1) is BIBS stable and take a triple $(\overline{u}, \Omega^a, \Omega^b)$. Assume that a positive δ exists such that the system is uniformly Lipschitz drift-observable in Ω_δ^b, and property $\mathcal{H}^0$ holds in $\Phi_{\overline{s}}(\Omega_\delta^b)$. Then, for any $\alpha > 0$, a gain matrix $K \in \mathbb{R}^{n\times q}$ for the observer (3.1) and positive constants μ_1, μ_2 and u_M exist such that if $\|u(t)\| \leq u_M$, $\forall t \geq 0$, $x(0) \in \Omega^a$ and

$$\hat{x}(0): \quad \|x(0) - \hat{x}(0)\| \leq \frac{\delta - \mu_2 b}{\mu_1}, \qquad \left(b = \sqrt{b_{\overline{L}}^2 + b_{\overline{H}}^2}\right) \tag{3.22}$$

then, provided that $\delta > \mu_2 b$, it follows

$$\|x(t) - \hat{x}(t)\| \leq \mu_1 e^{-\alpha t}\|x(0) - \hat{x}(0)\| + \mu_2 b \quad \forall t \geq 0. \tag{3.23}$$

Proof: Let $\gamma_{\Phi_{\overline{s}}}$ be the Lipschitz constant of $\Phi_{\overline{s}}$ in Ω_δ^b and $\gamma_{\Phi_{\overline{s}}^{-1}}$ be the Lipschitz constant of $\Phi_{\overline{s}}^{-1}$ in $\Phi_{\overline{s}}(\Omega_\delta^b)$. Let $\gamma_{\overline{L}}$, $b_{\overline{L}}$, and $\gamma_{\overline{H}}$, $b_{\overline{H}}$ be the constants of the ALG functions $\overline{L}_{\overline{s}}(z)$ and $\overline{H}_{\overline{s},\overline{r}}(z)$ in $\Phi_{\overline{s}}(\Omega_\delta^b)$. The map $z = \Phi_{\overline{s}}(x)$ is a coordinate transformation in Ω_δ^b for both system (1.1) and observer (3.1). Their expressions in z-coordinates are given by (2.15) and (3.4), and the error dynamics are governed by (3.5), where v_1 and v_2 defined in (3.6) are such that $\forall z, \hat{z} \in \Phi_{\overline{s}}(\Omega_\delta^b)$

$$\begin{aligned} \|v_1(z,\hat{z})\|^2 &\leq \gamma_{\overline{L}}^2\|e_z\|^2 + b_{\overline{L}}^2 \\ \|v_2(z,\hat{z})\|^2 &\leq \gamma_{\overline{H}}^2\|e_z\|^2 + b_{\overline{H}}^2. \end{aligned} \tag{3.24}$$

Consider now inequality (2.17) for a given positive α and with $\gamma^2 = \gamma_{\overline{L}}^2 + \gamma_{\overline{H}}^2$. Thanks to Theorem 2.7, a positive $\tilde{u}_M$ exists such that setting $\rho = \tilde{u}_M$ the inequality admits a stabilizing solution pair (K, P). Now, set $u_M = \min(\tilde{u}_M, \overline{u})$, use the gain K in the observer, and follow the same lines of the proof of Theorem 3.1. Use matrix P solution of (2.17) to define the positive definite function $\nu = e_z^{\mathrm{T}}P^{-1}e_z$ of the observation error e_z (3.8). The time derivative of ν along the error trajectory satisfies the inequality

$$\begin{aligned} \dot{\nu}(z,\hat{z}) \leq\, & e_z^{\mathrm{T}}P^{-1}\Big((A_{\overline{s}} - KC_{\overline{s}})P + P(A_{\overline{s}} - KC_{\overline{s}})^{\mathrm{T}} \\ & + B_{\overline{s}}B_{\overline{s}}^{\mathrm{T}} + u_M^2 F_{\overline{s},\overline{r}}F_{\overline{s},\overline{r}}^{\mathrm{T}} + \gamma^2 P^2\Big)P^{-1}e_z \\ & + b_{\overline{L}}^2 + b_{\overline{H}}^2 \end{aligned} \tag{3.25}$$

as long as z and $\hat{z}$ are in $\Phi_{\overline{s}}(\Omega_\delta^b)$. From this and from (2.17), the following inequality can be derived:

$$\dot{\nu}(z,\hat{z}) \leq -2\alpha\nu(e_z) + b_{\overline{L}}^2 + b_{\overline{H}}^2, \quad \text{for } z, \hat{z} \in \Phi_{\overline{s}}(\Omega_\delta^b). \tag{3.26}$$

Now, it is important to prove that $z(t)$ and $\hat{z}(t)$ remain in $\Phi_{\overline{s}}(\Omega_\delta^b)$ for all $t \geq 0$. From the BIBS assumption and being $x(0) \in \Omega_a$ and for all $t \geq 0$ $\|u(t)\| \leq u_M \leq \overline{u}$, then $x(t) \in \Omega^b$. *A fortiori*, $x(t) \in \Omega_\delta^b$ and $z(t) \in \Phi_{\overline{s}}(\Omega_\delta^b)$ for all $t \geq 0$. Now, choose $\mu_1 = \gamma_{\Phi_{\overline{s}}}\gamma_{\Phi_{\overline{s}}^{-1}}\sqrt{\mathrm{cn}(P)}$ and $\mu_2 = \gamma_{\Phi_{\overline{s}}^{-1}}\sqrt{\|P\|/(2\alpha)}$. Note that $\mu_1 > 1$, so that from assumption (3.22), also $\hat{x}(0) \in \Omega_\delta^a$ and $\hat{z}(0) \in \Phi_{\overline{s}}(\Omega_\delta^b)$. It remains to show that $\hat{x}(t) \in \Omega_\delta^b$ $\forall t \geq 0$ (and, hence, $\hat{z}(t) \in \Phi_{\overline{s}}(\Omega_\delta^b)$). This is made by proving that a finite exit time t_δ for $\hat{x}$ from the set Ω_δ^b does not exist. For, let an exit time $t_\delta < \infty$ exist. By definition, t_δ is such

that $\hat{x}(t_\delta) \in \partial\Omega_\delta^b$, and $\hat{x}(t) \in \Omega_\delta^b \backslash \partial\Omega_\delta^b$ for $t \in [0, t_\delta)$. From (3.26), the following inequality is obtained for $t \in [0, t_\delta]$

$$\|e_z(t)\|^2 \leq \mathrm{cn}(P) e^{-2\alpha t} \|e_z(0)\|^2 + \left(1 - e^{-2\alpha t}\right) \frac{\|P\|}{2\alpha} \left(b_L^2 + b_H^2\right). \tag{3.27}$$

From this, inequality (3.23) is readily proven for $t \in [0, t_\delta]$. It remains to prove that a finite exit time t_δ does not exist. Thanks to assumption (3.22), inequality (3.23) in $t \in (0, t_\delta]$ implies $\|e_x(t)\| < \delta$, and since $x(t_\delta) \in \Omega^b$, it follows that $\hat{x}(t_\delta) \in \Omega_\delta^b \backslash \partial\Omega_\delta^b$. This contradicts the fact that t_δ is a finite exit time and, therefore, (3.23) holds for all $t \geq 0$. ■

The semiglobal version of Theorem 3.4 is presented below. The proof is obtained following the same steps of Theorem 3.4 and Theorem 3.8 and is not reported for brevity.

Theorem 3.9: Assume that system (1.1) is BIBS stable and take a triple $(\overline{u}, \Omega^a, \Omega^b)$. Assume that a positive δ exists such that the system is ULDO in Ω_δ^b and that the vector observation relative degree $\overline{r}$ is $\overline{r} \geq \overline{s}$ in Ω_δ^b. Moreover, assume that property $\mathcal{H}^0$ holds in $\Phi_{\overline{s}}(\Omega_\delta^b)$ and that the input is bounded by a constant u_M $(\|u(t)\| \leq u_M, \forall t \geq 0)$. Then, for any $\alpha > 0$, a gain matrix $K \in \mathbb{R}^{n \times q}$ for the observer (3.1) and positive constants μ_1, μ_2 exist such that if $x(0) \in \Omega^a$ and

$$\hat{x}(0): \quad \|x(0) - \hat{x}(0)\| \leq \frac{\delta - \mu_2 b}{\mu_1} \qquad \left(b = \sqrt{b_{\overline{L}}^2 + b_{\overline{H}}^2}\right) \tag{3.28}$$

then, provided that $\delta > \mu_2 b$, inequality (3.23) follows. ■

Remark 3.10: Also for Theorem 3.9, if the assumption of ALG nonlinearities is relaxed to Lipschitz nonlinearities (assumption $\mathcal{H}^1$ is made instead of $\mathcal{H}^0$), convergence to zero of the observation error is obtained (the constant b is zero in the theorem statement). ■

IV. Comments on the Observer Implementability

The observer presented in this paper applies to a quite general class of nonlinear systems (drift-observable nonlinear systems with ALG or Lipschitz nonlinearities and with a suitably bounded input). The computation of matrix $Q_{\overline{s}}(\hat{x})$ in the observer (3.1) is straightforward: it requires repeated Lie derivatives and can be automatically performed by symbolic manipulators of equations. The choice of the gain matrix K is less direct: K should be a solution of the H_∞ Riccati-like inequality (2.17), in which the knowledge of the constants of the ALG (or Lipschitz) functions $\overline{L}_{\overline{s}}(z)$ and $\overline{H}_{\overline{s}, \overline{r}}(z)$ is required. On the other hand, in practical applications, the computation of such constants is not an easy task. However, the proofs of the theorems of existence of solutions pairs for the H_∞ Riccati-like inequality provide a strategy for the choice of the gain K. As pointed out in Remark 2.11, solutions of the inequality can be found adopting the choice $(K(w), P(w))$ by choosing a sufficiently large parameter w. This fact classifies the proposed observer as a *high-gain observer*. In practical applications, the tuning of the parameter w in the gain $K(w)$ can be made by evaluating the observer behavior through computer simulation. In the case when the needed ALG (or Lipschitz) constants are known, K can be found by solving the H_∞ Riccati-like inequality. Note that in the semiglobal observer, the size of the region of convergence depends also on the condition number of matrix P, and, therefore, it is advisable to choose, among the solution pairs (K, P), one with a small $\mathrm{cn}(P)$.

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