QUALITATIVE BEHAVIOR OF SOLUTIONS

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1 - PRELIMINARIES

1.1 - EXISTENCE AND UNIQUENESS OF SOLUTIONS

Existence and uniqueness of solutions are clearly the basic properties an Ordinary Differential Equations (ODE) model is required to satisfy in order to properly represent the physical system it aims to model. Before getting into the details, let us recall some mathematical definitions.

Definition 1.1.1 *A function* $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ *is:*

i) locally Lipschitz on the open set $D \subseteq \mathbb{R}^n$ *if for any* $x_0 \in D$ *there exists a neighborhood* $I_\rho(x_0) \subset D$ *such that, for some positive constant* L_0*:*

$$\|f(x_1) - f(x_2)\| \leq L_0 \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in I_\rho(x_0); \tag{1.1.1}$$

ii) Lipschitz on the open set $D \subseteq \mathbb{R}^n$ *if there exists a positive constant* L*:*

$$\|f(x_1) - f(x_2)\| \leq L \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in D. \tag{1.1.2}$$

iii) globally Lipschitz if it is Lipschitz on $D \equiv \mathbb{R}^n$*.*

Remark 1.1.2. Not all the *locally Lipschitz* functions are *Lipschitz* on the same domain D, since to this aim the Lipschitz condition needs to hold *uniformly* on D. However, it can be proven that a locally Lipschitz function on $D \subseteq \mathbb{R}^n$ is Lipschitz on every *compact* subset of D. •

Remark 1.1.3. Consider the Lipschitz condition for scalar functions $f : \mathbb{R} \mapsto \mathbb{R}$, which can be written as:

$$\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \leq L. \tag{1.1.3}$$

It means that:

i) if a function has a *discontinuity* in a given point x_0 (e.g. of the type: $f(x_0^-) \neq f(x_0^+)$), it cannot be locally Lipschitz on a domain which includes x_0;

ii) if a continuous function has an *infinite slope* in a given point x_0 (e.g. $f(x) = x^{1/3}$ for $x_0 = 0$), it cannot be locally Lipschitz on a domain which includes x_0;

iii) if a continuous function has a *bounded derivative* on a domain D (that means: $|f'(x)| \leq k$ for any $x \in D$), then it is Lipschitz on the domain. •

This last consideration can be properly extended to the general vectorial case.

Lemma 1.1.4. *Let* $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ *be continuous on the open set* $D \subseteq \mathbb{R}^n$ *and have continuous derivative* df/dx *on the same domain* D*. Then* $f(\cdot)$ *is locally Lipschitz on* D*. Moreover, if* df/dx *is uniformly bounded on a convex subset:*

$$\left\| \frac{df}{dx}(x) \right\| \leq k, \qquad \forall x \in W \subseteq D, \tag{1.1.4}$$

then $f(\cdot)$ *is Lipschitz on* W *with Lipschitz constant* $L = k$*.*

We are now in position to properly state the "*existence and uniqueness problem*" for an ODE model. Consider the time-invariant, nonlinear system:

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(t_0) = x_0, \qquad x(t) \in \mathbb{R}^n, \qquad f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1.1.5}$$

Since the system is time-invariant, the time evolution $x(t)$ corresponding to the unique solution (if any) of the Cauchy problem formalized by (1.1.5) depends only of the time difference $t - t_0$ instead of the two distinct time instants t_0 and t. For this reason, in the following, the initial time instant t_0 will be set equal to 0, with no loss of generality and the evolution $x(t)$ will be associated to a function $\varphi : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that:

$$x(t) = \varphi(t, x_0) \tag{1.1.6}$$

is the unique solution (if any) of the Cauchy problem (1.1.5) with $t_0 = 0$. φ is the **state-transition map**. In order to be meaningful, the state-transition map is required to verify the properties of **consistency** and **semigroup**. The consistency property claims that:

$$\varphi(0, x) = x, \qquad \forall x \in \mathbb{R}^n. \tag{1.1.7}$$

Indeed, when $t \mapsto 0^+$ the time evolution $x(t)$ needs to converge to the initial state x_0. As far as the semigroup property, it claims that:

$$\varphi\big(t - t_1, \varphi(t_1, x_0)\big) = \varphi(t, x_0), \qquad \forall x_0 \in \mathbb{R}^n, \qquad 0 \le t_1 \le t. \tag{1.1.8}$$

Note that $\varphi(t, x_0)$ is the time evolution starting from the initial time $t_0 = 0$ and initial state x_0, whilst $\varphi\big(t - t_1, \varphi(t_1, x_0)\big)$ is the time evolution starting from the initial time $t_1 > 0$ and initial state $x_1 = \varphi(t_1, x_0)$ reached by the system when starting from the initial time $t_0 = 0$ and initial state x_0. The semigroup property claims both the state to be equal.

Theorem 1.1.5. (Local Existence and Uniqueness) *Consider the time-invariant, nonlinear system defined in* (1.1.5) *with* $t_0 = 0$. *If* $f(\cdot)$ *is Lipschitz on a neighborhood* $I_\rho(x_0)$ *of the initial state, then there exists a* $\Delta t > 0$ *such that a unique solution* $x(t) = \varphi(t, x_0)$ *exists for* $t \in [0, \Delta t]$.

Example 1.1.6. Consider the case: $f(x) = x^{1/3}$, $x_0 = 0$. As stated in Remark 1.1.3, $f(\cdot)$ is not Lipschitz in the neighborhood of $x_0 = 0$. Indeed, it does not admit a unique solution since:

$$x(t) \equiv 0 \qquad \text{and} \qquad x(t) = \left(\frac{2t}{3}\right)^{3/2} \tag{1.1.9}$$

are both solutions of the problem. •

Remark 1.1.7. The local Lipschitz condition is not restrictive. Nevertheless, it does not allow to say how far the solution $x(t)$ may be extended on the time interval. In other words, it does not tell about the length of Δt. •

Example 1.1.8. Consider the case: $f(x) = x^2$ with a generic initial condition $x(0) = x_0$. It clearly comes that $f(\cdot)$ is locally Lipschitz on $\mathbb{R}^n$. Indeed, the unique solution is:

$$x(t) = \frac{x_0}{1 - t x_0}, \quad t \in [0, \Delta t], \qquad \text{for some} \quad \Delta t > 0. \tag{1.1.10}$$

However:

i) if $x_0 \le 0$, the solution can be extended up to infinity (i.e. for $t \in [0, +\infty)$);
ii) if $x_0 > 0$, the solution can be extended up to a *finite escape time* (i.e. for $t \in [0, 1/x_0)$). •

Theorem 1.1.9 (Global Existence and Uniqueness). *Consider the time-invariant, nonlinear system defined in* (1.1.5) *with* $t_0 = 0$. *If* $f(\cdot)$ *is globally Lipschitz, then there exists a unique solution* $x(t) = \varphi(t, x_0)$ *on a time interval* $t \in [0, T]$, *for any* $T > 0$.

Example 1.1.10. It has to be stressed that Theorem 1.1.9 provides a sufficient condition, which is somewhat rare to obtain. For instance, the cases $f(x) = x^2$ of Example 1.1.8 does not fulfill it. It means that its solution *may not* be indefinitely extended (and this happens, for instance, for $x_0 > 0$), but *does not* prevent it at all (as it happens for $x_0 \leq 0$). •

Example 1.1.11. Linear functions $f(x) = Ax$ are globally Lipschitz, therefore time-invariant, linear systems of the type:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \qquad x(t) \in I\!R^n, \qquad A \in I\!R^{n\times n} \tag{1.1.11}$$

always admit a unique solution on $[0,T]$ for any $T > 0$ (and prevent finite escape times). •

1.2 - CONTINUOUS DEPENDENCE ON INITIAL CONDITION AND PARAMETERS

If existence and uniqueness is a "*conditio sine qua non*" a mathematical model can be usefully considered to represent the underlying physics, continuous dependence on the initial condition and on the function $f(\cdot)$ are of primary importance with respect to all other properties, since we want small variations for the evolution $x(t)$ according to small perturbations of the initial conditions or of the model parameters. To this aim the following definitions are required.

Definition 1.2.1. *Consider the time-invariant, nonlinear system defined in* (1.1.5) *with* $t_0 = 0$*. Assume it admits a unique solution* $x(t) = \varphi(t, x_0)$ *defined on* $t \in [0,T]$*. This solution depends continuously on the initial state* x_0 *if:*

$$\forall \varepsilon > 0, \quad \exists \delta > 0: \quad \forall \tilde{x}_0 \in I_\delta(x_0) \;\Rightarrow\; \begin{cases} \dot{x}(t) = f\big(x(t)\big), \quad x(0) = \tilde{x}_0 \quad \textit{admits a unique solution on } [0,T], \\ \|\varphi(t,x_0) - \varphi(t,\tilde{x}_0)\| < \varepsilon, \quad \forall t \in [0,T]. \end{cases} \tag{1.2.1}$$

Remark 1.2.2. According to Definition 1.2.1, continuous dependence on the initial state means that, no matter how close (i.e. how small is ε) to the original evolution $\varphi(t,x_0)$ we claim the perturbed evolution $\varphi(t,\tilde{x}_0)$ be, we can always choose a perturbation of the initial state close enough to the original initial state x_0 such that there exists a unique solution of the perturbed system which satisfies the constraint of closeness to the original solution. •

As far as the continuous dependence on $f(\cdot)$, the problem will be reduced to the continuous dependence on a parameter vector θ characterizing $f(\cdot)$.

Definition 1.2.3. *Consider the time-invariant, nonlinear system:*

$$\dot{x}(t) = f\big(x(t),\theta\big), \qquad x(0) = x_0, \qquad x(t) \in I\!R^n, \qquad \theta \in \Theta \subseteq I\!R^p, \qquad f: I\!R^n \times I\!R^p \mapsto I\!R^n. \tag{1.2.2}$$

Assume it admits a unique solution $x(t) = \varphi(t,x_0,\theta)$ *defined on* $t \in [0,T]$*. This solution depends continuously on the parameter vector* θ *if:*

$$\forall \varepsilon > 0, \quad \exists \delta > 0: \quad \forall \tilde{\theta} \in I_\delta(\theta) \;\Rightarrow\; \begin{cases} \dot{x}(t) = f\big(x(t),\tilde{\theta}\big), \quad x(0) = x_0 \quad \textit{admits a unique solution on } [0,T], \\ \|\varphi(t,x_0,\theta) - \varphi(t,x_0,\tilde{\theta})\| < \varepsilon, \quad \forall t \in [0,T]. \end{cases} \tag{1.2.3}$$

Remark 1.2.4. Like in Remark 1.2.2, according to Definition 1.2.3, continuous dependence on the parameter vector means that, no matter how close (i.e. how small is ε) to the original evolution $\varphi(t,x_0,\theta)$ we claim the perturbed evolution $\varphi(t,x_0,\tilde{\theta})$ be, we can always choose a perturbation of the vector parameter close enough to the original value θ such that there exists a unique solution of the perturbed system which satisfies the constraint of closeness to the original solution. •

Definitions 1.2.1 and 1.2.3 are both taken into account in the following Theorem stating the hypotheses according to which system (1.2.2) ensures continuous dependence on both the initial condition and the parameter vector.

Theorem 1.2.5 (Continuous Dependence on Initial Condition and Parameters). *Consider the time-invariant, nonlinear system defined in* (1.2.2)*. Let $f(\cdot,\cdot)$ be continuous with respect to θ on a given neighborhood $I_\rho(\theta)$ and Lipschitz with respect to x on an open set $D \subseteq I\!\!R^n$ containing the initial condition x_0. Suppose $x(t) = \varphi(t, x_0, \theta)$, $t \in [0, T]$ is the unique solution of* (1.2.2) *and belongs to D for all $t \in [0, T]$. Then:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \begin{cases} \forall \tilde{x}_0 \in I_\delta(x_0) \\ \forall \tilde{\theta} \in I_\delta(\theta) \end{cases} \Rightarrow \begin{cases} \dot{x}(t) = f\big(x(t), \theta\big), \quad x(0) = \tilde{x}_0 \quad \textit{admits a unique solution on } [0,T], \\ \|\varphi(t, x_0, \theta) - \varphi(t, \tilde{x}_0, \tilde{\theta})\| < \varepsilon, \quad \forall t \in [0, T]. \end{cases} \tag{1.2.4}$$

Remark 1.2.6 As previously stated in Example 1.1.11, linear functions are globally Lipschitz, that means solutions of linear systems continuously depend on the initial state. •

2 - EQUILIBRIUM POINTS AND STABILITY

2.1 - EQUILIBRIUM POINTS

Assuming an ODE model admits a unique solution compatible with the initial state, the explicit solution could be very hard to compute and, often, may not exist in an analytical closed form, unless particular cases such as linear models. The *qualitative* analysis of solutions of an ODE model (which dates back to the pioneering works of Poincaré around 1880) gives useful insights on the real behavior of the solution, without involving a *quantitative* analysis based on the analytical/numerical methodologies required to compute explicitely the solution.

Definition 2.1.1. *Consider the time-invariant, nonlinear system:*

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0, \qquad x(t) \in I\!\!R^n, \qquad f : I\!\!R^n \mapsto I\!\!R^n, \tag{2.1.1}$$

with $x(t) = \varphi(t, x_0)$ the unique solution associated to (2.1.1)*. $x_e \in I\!\!R^n$ is an equilibrium point if:*

$$\varphi(t, x_e) = x_e, \qquad \forall t \geq 0. \tag{2.1.2}$$

Remark 2.1.2. According to Definition 2.1.1, it comes that if the initial state is an equilibrium point, then there is no motion, that means the trajectory reduces trivially to the equilibrium point. Therefore a way to find the equilibrium points is to solve the algebraic nonlinear system: $f(x) = 0$, since the solutions make the time derivative vanish. As a matter of fact, it comes that:

- there could be no equilibrium points at all ($f(x) = 0$ has no solutions);
- there could be a unique equilibrium point ($f(x) = 0$ admits a unique solution);
- there could be a finite number of isolated equilibrium points ($f(x) = 0$ admits a finite number of solutions);
- there could be an infinite number (countable or uncountable) of equilibrium points ($f(x) = 0$ admits infinite solutions). •

Remark 2.1.3. In case of linear systems $\dot{x}(t) = Ax(t)$, the equilibrium points are the solutions of $Ax = 0$, therefore they consist of the null-space of matrix A. It means:
– the origin is always an equilibrium point;
– if $\mathrm{rank}(A) = n$, the origin is the unique equilibrium point;
– if $\mathrm{rank}(A) = r < n$, there exist ∞^{n-r} (uncountable) equilibrium points;
– there can never be a finite (neither a countable) number of isolated equilibrium points.

2.2 - STABILITY

Stability theory allows to understand what happens in case of perturbations. In this framework we will consider perturbations of the equilibrium points: how close to the equilibrium point the motion is kept according to a perturbation of the initial state from the equilibrium.

Definition 2.2.1. *Consider the time-invariant nonlinear system defined in* (2.1.1)*. The equilibrium point x_e is stable if:*

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|x_0 - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \varepsilon, \quad \forall t \geq 0. \tag{2.2.1}$$

Remark 2.2.2. The stability definition is related to the capability of maintaining arbitrarily bounded the state evolution according to an initial state perturbation occurring in a suitably chosen neighborhood of the equilibrium state. In Fig. 2.2.1 a graphical interpretation for a second-order system is provided where Euclidean norms are considered, so that neighborhoods are circles. Then, Definition 2.2.1 can be stated as follows: *whatever arbitrarily small is chosen the radius ε of a neighborhood of the equilibrium point x_e, it must be possible to find a smaller radius δ such that, whatever is chosen the initial state x_0 inside the inner neighborhood $I_\delta(x_e)$, the time evolution $x(t)$ is kept constrained within the outer neighborhood $I_\varepsilon(x_e)$.* •

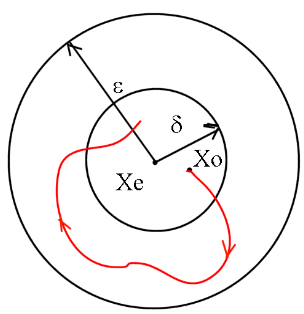

Fig. 2.2.1 - Definition of stability: graphical interpretation for second-order systems.

Lemma 2.2.3. *Consider a linear ODE model. In case of many infinite equilibrium points, the stability of a given equilibrium point implies and is implied by the stability of the origin.*

Proof. Let $\dot{x}(t) = Ax(t)$ be the linear system under investigation, and let $x_e \neq 0$ a nontrivial equilibrium point (i.e. $Ax_e = 0$). If x_e is stable, then condition (2.2.1) holds true. Now consider the displacement $z(t) = x(t) - x_e$, whose dynamics is described by:

$$\dot{z}(t) = \dot{x}(t) = Ax(t) = Ax(t) - Ax_e = Az(t). \tag{2.2.2}$$

That is: $z(t)$ is described by the same linear ODE model of $x(t)$. Thus, the stability condition for the origin written in the z-coordinates:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|z_0\| < \delta \quad \Longrightarrow \quad \|z(t)\| < \varepsilon, \quad \forall t \geq 0 \tag{2.2.3}$$

is the same of (2.2.1), referred to the stability of x_e: if (2.2.1) holds true, also (2.2.3) holds true and conversely. ♢

Remark 2.2.4. According to Lemma 2.2.3, we will talk about the *stability of a linear system* rather than the *stability of the origin of a linear system.* •

Definition 2.2.5. *Consider the time-invariant nonlinear system defined in* (2.1.1)*. The equilibrium point x_e is*

– *locally attractive if:*

$$\exists \eta > 0 : \quad \|x_0 - x_e\| < \eta \quad \Longrightarrow \quad \|x(t) - x_e\| \mapsto 0. \tag{2.2.4}$$

– *globally attractive if:*

$$\forall x_0 \in I\!R^n, \quad it\ is: \quad \|x(t) - x_e\| \mapsto 0. \tag{2.2.5}$$

Remark 2.2.6. By definition, attractivity can occur only if the equilibrium point x_e is isolated since, otherwise, whatever small δ is chosen, there will always be many infinite equilibrium points in $I_\rho(x_e)$: if one of them is chosen as initial state, there will be no motion and, therefore, no convergence to x_e. •

It has to be stressed that an equilibrium point can be attractive without necessarily being stable. The following example allows to better understand the statement.

Example 2.2.7. Consider the ODE model:

$$\begin{cases} \dot{x}_1 = \dfrac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\big(1 + (x_1^2 + x_2^2)\big)}, \\ \\ \dot{x}_2 = \dfrac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\big(1 + (x_1^2 + x_2^2)\big)}. \end{cases} \tag{2.2.6}$$

It can be proven that the origin is an equilibrium point, which is attractive but not stable. Indeed, wherever starting in the second quadrant of the plane (even when starting very close to the equilibrium point), the evolution approaches a curve which lies on a finite distance from the origin before it converges to it (see Fig. 2.2.2). We cannot constrain the evolution on a neighborhood which does not contain such a curve, that means: no stability! •

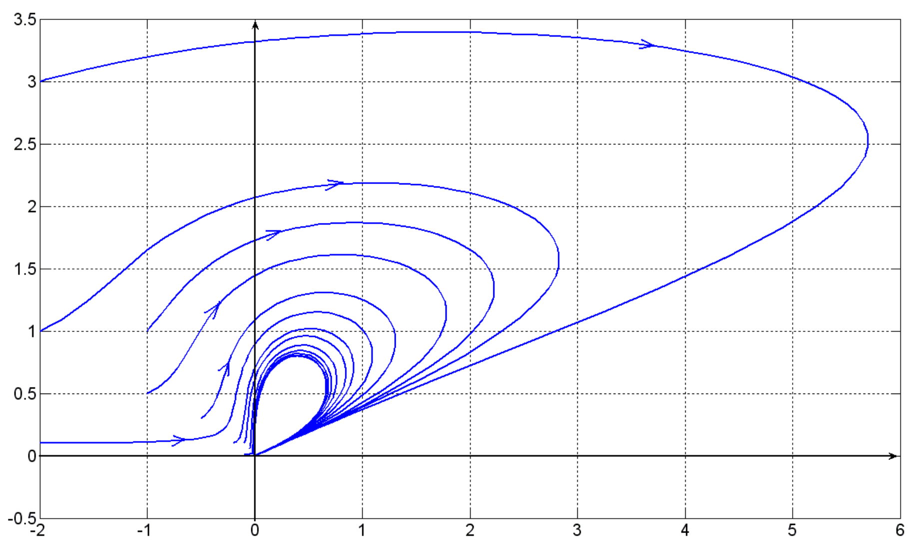

Fig. 2.2.2 - Trajectories referred to system (2.2.6).

Definition 2.2.8. *Consider the time-invariant nonlinear system defined in* (2.1.1)*. The equilibrium point x_e is locally/globally asymptotically stable if it is stable and locally/globally attractive.*

Definition 2.2.9. *Consider the time-invariant nonlinear system defined in* (2.1.1)*. The equilibrium point x_e is exponentially stable if:*

$$\exists \alpha > 0 : \quad \forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|x_0 - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \varepsilon \cdot e^{-\alpha t}. \tag{2.2.7}$$

Remark 2.2.10. Note that exponential stability implies local asymptotic stability since, according to (2.2.7), it is:

$$\|x(t) - x_e\| < \varepsilon \cdot e^{-\alpha t} < \varepsilon \qquad \text{and} \qquad \|x(t) - x_e\| < \varepsilon \cdot e^{-\alpha t} \mapsto 0. \tag{2.2.8}$$

Moreover, the exponential stability ensures a "*faster-than-exponential*" convergence to zero of the displacement $x(t) - x_e$. •

Lemma 2.2.11. *Consider a linear ODE model $\dot{x}(t) = Ax(t)$. Then only the origin can be asymptotically stable, if it is the only equilibrium point.*

Proof. The proof follows from the consideration that the set of the equilibrium points of a linear system can either reduce to the only origin (if $\text{rank}(A) = n$) or be constituted of a subspace of the state space (if $\text{rank}(A) < n$), see Remark 2.1.3. For instance, if we consider a second-order system with $\text{rank}(A) = 1 < n = 2$, the equilibrium points constitute a right line passing through the origin. Thus, the Lemma is proven according to Remark 2.2.6. ♢

3 - LINEAR SYSTEMS

3.1 - SPECTRAL DECOMPOSITION OF A SQUARE MATRIX

Linear ODE models allow to analytically compute in a closed form the explicit solution of a Cauchy problem. Let us consider a linear ODE model of the type:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in I\!R^n, \qquad A \in I\!R^{n \times n}, \tag{3.1.1}$$

and assume that matrix A has n distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, to which the following eigenvectors are associated: $\{u_1, \ldots, u_n\}$ such that:

$$Au_i = \lambda_i u_i. \tag{3.1.2}$$

These eigenvectors constitute a base for the state space, that means the following matrix:

$$U = [u_1 \ \cdots \ u_n] \tag{3.1.3}$$

is nonsingular. Thus, it is easy to verify that:

$$AU = A[u_1 \ \cdots \ u_n] = [Au_1 \ \cdots \ Au_n] = [\lambda_1 u_1 \ \cdots \ \lambda_n u_n] = [u_1 \ \cdots \ u_n] \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} = U\Lambda, \tag{3.1.4}$$

where Λ is the diagonal matrix in $I\!R^{n \times n}$ having the n eigenvalues on its main diagonal. From eq.(3.1.4) it follows that A and Λ are *similar* matrices, that means there exists a coordinate transformation matrix U such that in the new coordinates $z(t) = U^{-1}x(t)$, matrix Λ plays the same role of matrix A for the x-dynamics. Indeed, by writing the z-dynamics:

$$\dot{z}(t) = U^{-1}\dot{x}(t) = U^{-1}Ax(t) = U^{-1}AUz(t) = \Lambda z(t). \tag{3.1.5}$$

Moreover, the rows of $V = U^{-1}$ are left-eigenvectors of matrix A, since:

$$U^{-1}A = VA = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} A = \begin{bmatrix} v_1^T A \\ \vdots \\ v_n^T A \end{bmatrix} \qquad \text{and} \qquad \Lambda U^{-1} = \Lambda V = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1^T \\ \vdots \\ \lambda_n v_n^T \end{bmatrix}. \tag{3.1.6}$$

In summary, matrix A can be written as:

$$A = U\Lambda V = [u_1 \ \cdots \ u_n] \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = \sum_{i=1}^{n} \lambda_i u_i v_i^T \tag{3.1.7}$$

and this decomposition is known as **spectral decomposition** of matrix A with all distinct eigenvalues.

Remark 3.1.1. Note that left-eigenvectors are orthogonal to (right)-eigenvectors when associate to different eigenvalues, that is:

$$v_i^T u_j = 0, \qquad i \neq j. \tag{3.1.8}$$

Moreover, in case matrix A is symmetric ($A = A^T$), it is:

$$Au_i = \lambda_i u_i \qquad \Longrightarrow \qquad u_i^T A^T = u_i^T A = \lambda_i u_i^T, \tag{3.1.9}$$

that means: left-eigenvectors are (right)-eigenvectors transposed. •

In case of multiple eigenvalues, a base of *generalized eigenvectors* can be written such that:

$$A = UJU^{-1}, \tag{3.1.10}$$

where J has the eigenvalues on the main diagonal, 1 or 0 on the superdiagonal, and zeros elsewhere.

Definition 3.1.2. *Given a square matrix $A \in \mathbb{R}^{n\times n}$, the exponential matrix e^A is defined as follows:*

$$e^A = I + A + \frac{A^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{3.1.11}$$

Remark 3.1.3. Note that Definition 3.1.2 is well posed, since:

$$\left\| e^A \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}. \tag{3.1.12}$$

It as to be stressed that the left-hand side of inequality (3.1.12) is the norm of a matrix (the exponential matrix), while the right-hand side is the exponential of a scalar (the norm of a matrix). •

In order to compute the exponential matrix, let us first compute the generic power of a matrix A, by suitably exploiting the spectral decomposition.

Lemma 3.1.4. *The k-th power ($k \geq 0$) of a matrix $A \in \mathbb{R}^{n\times n}$ is given by:*

$$A^k = U\Lambda^k U^{-1} \quad (n \text{ distinct eigenvalues}), \qquad A^k = UJ^kU^{-1} \quad (\text{multiple eigenvalues}), \tag{3.1.13}$$

where U is the eigenvectors matrix defined in (3.1.3), *Λ is the eigenvalues matrix defined in* (3.1.4) *and J is the counterpart of Λ in case of multiple eigenvalues.*

Proof. The Lemma is proven by induction. Assume n distinct eigenvalues. The first identity in (3.1.13) is readily verified for $k = 0$, since:

$$A^0 = I_n, \qquad \text{and} \qquad U\Lambda^0 U^{-1} = UU^{-1} = I_n. \tag{3.1.14}$$

Now assume $A^k = U\Lambda^k U^{-1}$ is true for an integer $k \geq 0$. Then, by using the spectral decomposition:

$$A^{k+1} = A \cdot U\Lambda^k U^{-1} = U\Lambda U^{-1} \cdot U\Lambda^k U^{-1} = U\Lambda \cdot \Lambda^k U^{-1} = U\Lambda^{k+1} U^{-1}. \tag{3.1.15}$$

The case of multiple eigenvalues is straightforward. $\diamondsuit$

Lemma 3.1.4 allows a closed-form computation for the exponential of matrix A.

Lemma 3.1.5. *In case of n distinct eigenvalues, the exponential of a matrix $A \in I\!R^{n\times n}$ is given by:*

$$e^A = \sum_{i=1}^{n} e^{\lambda_i} u_i v_i^T, \tag{3.1.16}$$

where λ_i, u_i, v_i^T, $i = 1, \ldots, n$ are the eigenvalues, the (right)-eigenvectors and the left-eigenvectors of the matrix.

Proof. According to the spectral decomposition, by using Definition 3.1.2 and Lemma 3.1.4, it comes:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{U\Lambda^k U^{-1}}{k!} = U \left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \right) U^{-1} = U e^{\Lambda} U^{-1}. \tag{3.1.17}$$

In fact, the exponential of matrix A is similar to the exponential of its diagonal form Λ, which is diagonal itself, since:

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & & O \\ & \ddots & \\ O & & e^{\lambda_n} \end{bmatrix} \tag{3.1.18}$$

which fact completes the proof. $\diamondsuit$

Remark 3.1.6. In case of multiple eigenvalues eq.(3.1.17) is modified as:

$$e^A = U e^J U^{-1}. \tag{3.1.19}$$

Unfortunately, eq.(3.1.19) cannot be easily transformed into the generalized version of eq.(3.1.16), since the exponential matrix e^J is not diagonal. $\bullet$

3.2 - EXPLICIT SOLUTIONS AND NATURAL MODES OF LINEAR SYSTEMS

The solution of a Cauchy problem, stated for a linear ODE model as defined in (3.1.1), is a linear transformation of the initial state x_0:

$$x(t) = \varphi(t, x_0) = \Phi(t)x_0, \qquad \Phi(t) \in I\!R^{n\times n}, \tag{3.2.1}$$

with matrix $\Phi(\cdot)$ known as the **state-transition matrix**.

Lemma 3.2.1. *Consider the linear system defined in* (3.1.1). *The related state-transition matrix $\Phi(\cdot)$ satisfies the following properties:*

i) $\Phi(0) = I_n$;

ii) $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2), \quad \forall t_1, t_2 \geq 0.$

Proof. Item i) is readily proven by suitably exploiting the consistency of the state-transition map:

$$x = \varphi(0, x) = \Phi(0)x, \qquad \forall x \in I\!R^n. \tag{3.2.2}$$

The arbitrariness of x necessarily fixes $\Phi(0) = I_n$.

Item ii) is proven by exploiting the semigroup property, since:

$$\varphi(t_1 + t_2, x_0) = \varphi\big(t_1, \varphi(t_2, x_0)\big), \qquad \forall t_1, t_2 \geq 0, \quad \forall x_0 \in I\!R^n. \tag{3.2.3}$$

According to the linearity of the system:

$$\varphi(t_1 + t_2, x_0) = \Phi(t_1 + t_2)x_0, \qquad \varphi\big(t_1, \varphi(t_2, x_0)\big) = \Phi(t_1)\varphi(t_2, x_0) = \Phi(t_1)\Phi(t_2)x_0. \tag{3.2.4}$$

The arbitrariness of the initial state x_0 completes the proof of item ii). $\diamondsuit$

Theorem 3.2.2. *Consider the linear system defined in* (3.1.1)*. The related state-transition matrix* $\Phi(\cdot)$ *obeys the following matricial ODE system:*

$$\frac{d\Phi}{dt} = A\Phi(t), \qquad \Phi(0) = I_n. \tag{3.2.5}$$

Proof. The initial condition trivially comes from the consistency property of Lemma 3.2.1. Moreover, by applying the time-derivative to the explicit solution $x(t) = \Phi(t)x_0$, it is:

$$\dot{x}(t) = \frac{d}{dt}\big[\Phi(t)x_0\big] = \frac{d\Phi}{dt}x_0 \qquad \text{and, by definition} \qquad \dot{x}(t) = Ax(t) = A\Phi(t)x_0. \tag{3.2.6}$$

The arbitrariness of x_0 completes the proof. $\diamondsuit$

Remark 3.2.3. It has to be stressed that, when computed for $t = 0$, eq.(3.2.5) states that $\dot{\Phi}(0) = A$. Moreover by applying the semigroup property to the time-derivative of the state transition matrix:

$$\begin{aligned} \frac{d\Phi}{dt} = \lim_{\Delta t \mapsto 0^+} \frac{\Phi(t + \Delta t) - \Phi(t)}{\Delta t} &= \lim_{\Delta t \mapsto 0^+} \frac{\Phi(\Delta t)\Phi(t) - \Phi(t)}{\Delta t} = \left(\lim_{\Delta t \mapsto 0^+} \frac{\big(\Phi(\Delta t) - I_n\big)}{\Delta t} \right) \Phi(t) = A\Phi(t) \\ &= \lim_{\Delta t \mapsto 0^+} \frac{\Phi(t)\Phi(\Delta t) - \Phi(t)}{\Delta t} = \Phi(t) \left(\lim_{\Delta t \mapsto 0^+} \frac{\big(\Phi(\Delta t) - I_n\big)}{\Delta t} \right) = \Phi(t)A, \end{aligned} \tag{3.2.7}$$

that means: $A\Phi(t) = \Phi(t)A$. $\bullet$

Theorem 3.2.4 *Consider the linear system defined in* (3.1.1)*. The related state-transition matrix* $\Phi(\cdot)$ *is the exponential matrix function:*

$$\Phi(t) = e^{At}. \tag{3.2.8}$$

Proof. The proof is achieved by showing that e^{At} is the solution to the matricial ODE system (3.2.5). Indeed, by definition:

$$e^{At}\Big|_{t=0} = \left[I_n + At + \frac{A^2t^2}{2} + \cdots \right]_{t=0} = I_n, \tag{3.2.9}$$

and:

$$\frac{d}{dt}\big[e^{At}\big] = \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right] = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A \sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = A \sum_{h=0}^{\infty} \frac{A^h t^h}{h!} = Ae^{At}. \tag{3.2.10}$$

$\diamondsuit$

The spectral decomposition of matrix A allows to write the explicit solution $x(t) = e^{At}x_0$ as follows. Assume at first to have n distinct eigenvalues. Since the eigenvectors constitute a base for the state space, the initial state itself can be written as:

$$x_0 = c_1 u_1 + \cdots + c_n u_n = U \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} c_i u_i, \tag{3.2.13}$$

with the vector $\alpha = [c_1 \ \ \ldots \ \ c_n]^T$ providing the coordinates of x_0 with respect to $\{u_1, \ldots, u_n\}$. Then, from (3.1.17):

$$x(t) = U e^{\Lambda t} U^{-1} U \alpha = U e^{\Lambda t} \alpha = \sum_{i=1}^{n} e^{\lambda_i t} c_i u_i. \tag{3.2.14}$$

The terms in the sum are known as **natural modes** of the system, and they contribute to the whole evolution only if the initial state *excites* them: x_0 needs to have a nontrivial i-th coordinate $c_i \neq 0$ in order to let the i-th mode $e^{\lambda_i t} c_i u_i$ be present in the time evolution. According to Remark 3.1.1, the i-th coordinate c_i is nontrivial if and only if the left-eigenvector v_i^T is not orthogonal to the initial state, that is:

$$v_i^T x_0 \neq 0. \tag{3.2.15}$$

If condition (3.2.15) holds true, we say that the initial state excites the i-th natural mode of the system.

According to the feature of the corresponding eigenvalue, a natural mode can be:

i) *aperiodic*, if λ is a real scalar: the time evolution is that of an exponential, $e^{\lambda t} cu$. In case of a negative eigenvalue we have an *asymptotically stable aperiodic* mode: the trajectory starts at a point of the rightline r containing vector u and converges to the origin with an infinite time along r, with no oscillations, in an exponential decreasing fashion, with rate $|\lambda|$ (see case A of Fig.3.2.1); in case of a positive eigenvalue we have an *unstable aperiodic* mode: the trajectory starts at a point of the rightline r containing vector u and escapes from the origin to infinity with an infinite time along r, with no oscillations, in an exponential increasing fashion, with rate λ (see case B of Fig.3.2.1); in case of $\lambda = 0$ we have a *stable aperiodic* mode: the trajectory is trivially the point of the initial state, which belongs to the rightline r containing vector u; there is no motion (see case C of Fig.3.2.1);

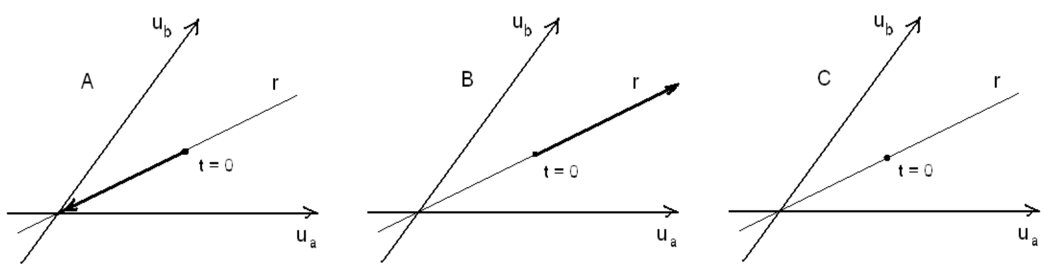

Fig.3.2.1 - Aperiodic natural modes

ii) *periodic*, if λ is a pure imaginary scalar, that is $\lambda = j\omega$. In this case it is useful to consider together the pair of complex conjugate modes (recall that matrix A has all real entries), so that:

$$\begin{aligned} e^{j\omega t} c_i u_i + e^{-j\omega t} c_i^{\star} u_i^{\star} &= \rho\Big(\big(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\big) u_a + j\big(e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\big) u_b \Big) \\ &= 2\rho\Big(\cos(\omega t+\theta) u_a - \sin(\omega t+\theta) u_b \Big), \end{aligned} \tag{3.2.16}$$

with $c_{1/2} = \rho e^{\pm j\theta}$ and $u_{1/2} = u_a \pm j u_b$. The time evolution is a periodic closed orbit on the subspace $\text{span}\{u_a, u_b\}$, with period $T = 2\pi/\omega$ (see Fig.3.2.2). The amplitude depends on the initial state, since ρ is the absolute value of the coordinate c;

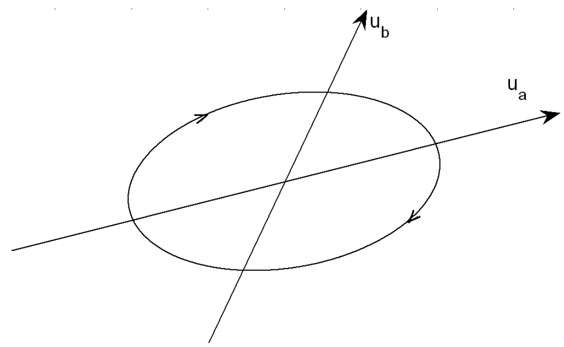

Fig.3.2.2 - Stable periodic natural mode

iii) *pseudoperiodic*, if λ is a generic complex eigenvalue, that is $\lambda = \alpha + j\omega$, $\alpha \neq 0$. Also in this case it is useful to consider together the pair of complex conjugate modes, so that:

$$\begin{aligned} e^{(\alpha+j\omega)t}c_i u_i + e^{(\alpha-j\omega)t}c_i^\star u_i^\star &= \rho e^{\alpha t}\Big(\big(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\big)u_a + j\big(e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\big)u_b\Big) \\ &= 2\rho e^{\alpha t}\Big(\cos(\omega t+\theta)u_a - \sin(\omega t+\theta)u_b\Big). \end{aligned} \tag{3.2.17}$$

with $c_{1/2} = \rho e^{\pm j\theta}$ and $u_{1/2} = u_a \pm j u_b$. In case of negative real part ($\alpha < 0$) we have an *asymptotically stable pseudoperiodic* mode: the trajectory is a spiral which belongs to the subspace span$\{u_a, u_b\}$, converging to the origin with an infinite time and exponential rate $|\alpha|$ (see case A of Fig.3.2.3); in case of positive real part ($\alpha > 0$) we have an *unstable pseudoperiodic* mode: the trajectory is a spiral which belongs to the subspace span$\{u_a, u_b\}$, escaping from the origin to infinity with an infinite time and exponential rate α (see case B of Fig.3.2.3).

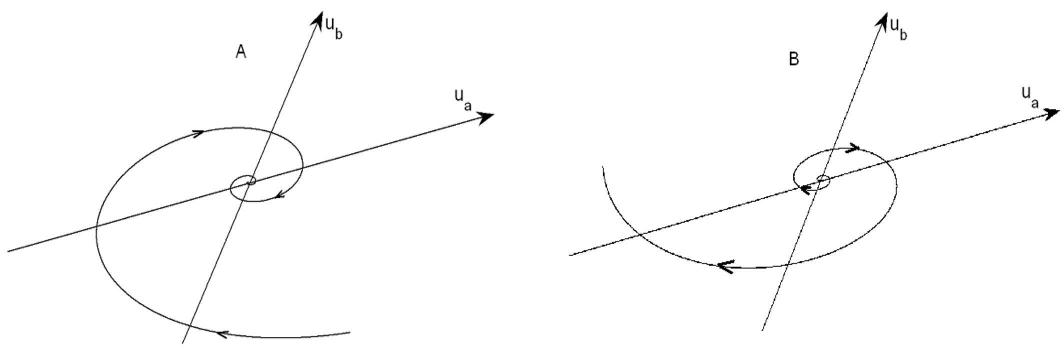

Fig.3.2.3 - Pseudoperiodic natural modes

In case of multiple eigenvalues, the time evolution (3.2.14) becomes:

$$x(t) = Ue^{Jt}U^{-1}U\alpha = Ue^{Jt}\alpha = \sum_{i=1}^{n} e^{\lambda_i t} p_i(t) u_i, \tag{3.2.18}$$

where $p_i(t)$ is a polynomial in t, whose degree is at most equal to $n-1$. Note that in eq.(3.2.18) it does happen that $\lambda_i = \lambda_j$ for some $i \neq j$. The computation of e^{Jt} is more cumbersome with respect to the case of n distinct eigenvalues, and will not be treated in details.

3.3 - STABILITY OF LINEAR SYSTEMS

Let us recall that for "*stability of a linear system*" we mean the stability of the origin, being the only equilibrium point which can be asymptotically stable and, in any case, its stability implies and is implied by the stability of all the other equilibrium point (if any). Moreover, in case of linear systems, there is a great advantage when dealing with stability: we know the explicit solution (eq.(3.2.14) for n distinct eigenvalues and eq.(3.2.18) for multiple eigenvalues)!

Theorem 3.3.1. *Consider the linear system defined in* (3.1.1). *It is asymptotically stable if, and only if, all its eigenvalues have (strictly) negative real part. In this case we have global, exponential stability.*

Proof. The case of all strictly negative real eigenvalues excludes the possibility of null eigenvalues, that means $\mathrm{rank}(A) = n$: the origin is the only equilibrium point. The attractivity readily comes by exploiting the explicit solutions (3.2.14) or (3.2.18). In both cases the evolution is the sum of decreasing exponentials (the case of n distinct eigenvalues) or functions bounded by decreasing exponentials (the case of multiple eigenvalues). Thus the attractivity is global with exponential rate. Stability also comes from the explicit solutions. For instance, consider the case of n distinct eigenvalues: whatever small is set ε, we can choose the initial condition in order to have the coefficients c_i small enough to constrain $x(t)$ within $I_\varepsilon(0)$ (recall that linear systems provide continuity with respect the initial condition, Remark 1.2.6). The proof is completed analogously for multiple eigenvalues. $\diamondsuit$

Remark 3.3.2. Even if there were only one eigenvalue with positive real part (and all the other with negative real part) the linear system would be unstable. •

Theorem 3.3.3. *Consider the linear system defined in* (3.1.1). *It is stable if all its non-multiple eigenvalues have non-positive real part, and all its multiple eigenvalues have (strictly) negative real part.*

Proof. Non-multiple eigenvalues with non-positive real part means stable and asymptotically stable natural modes, whose evolution can be constrained in any small neighborhood of the origin, due to the continuity of the solution with respect to the initial state. In case of multiple eigenvalues, the hypothesis of strictly negative eigenvalues prevents the case of elements in (3.2.18) with an eigenvalue λ_i with null real part and a polynomial $p_i(t)$ with degree ≥ 1, which would make the evolution diverge to infinity. $\diamondsuit$

Remark 3.3.4. Theorem 3.3.3 provides a sufficient condition. Indeed, there are cases for which multiple eigenvalues with null real part may produce stability (i.e. the degree of the corresponding polynomial $p_i(t)$ in (3.2.18) is $= 0$). However, these cases are not treated here. •

According to Theorems 3.3.1 and 3.3.3, the investigation of stability for linear systems reduces to the analysis of the sign of the real part of the eigenvalues, whose explicit computation in the general case is not trivial, unless we are considering a second-order system. Indeed the eigenvalues are the roots of the *characteristic equation*:

$$d(\lambda) = \det(\lambda I_n - A), \tag{3.3.1}$$

which is an n-th degree polynomial equation. The following criterion allows to answer to the stability problem without explicitely computing the eigenvalues.

Theorem 3.3.6 (Routh-Hurwitz Criterion). *Let*

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0, \tag{3.3.2}$$

an n-th degree polynomial ($\alpha_n \neq 0$). Consider Table 3.3.1, with:

i) $a_i^0 = \alpha_i, \qquad i = n, n-2, n-4, \ldots;$

ii) $a_i^1 = \alpha_i, \qquad i = n-1, n-3, n-5, \ldots;$

iii) $a_i^j = -\dfrac{b_i^j}{a_{n+1-j}^{j-1}}, \quad b_i^j = \begin{vmatrix} a_{n+2-j}^{j-2} & a_i^{j-2} \\ a_{n+1-j}^{j-1} & a_{i-1}^{j-1} \end{vmatrix}, \quad j = 2, 3, \ldots, n, \quad i = n-j, n-j-2, \cdots.$

$$
\begin{array}{c|ccccc}
n & a_n^0 & a_{n-2}^0 & a_{n-4}^0 & a_{n-6}^0 & \cdots \\
n-1 & a_{n-1}^1 & a_{n-3}^1 & a_{n-5}^1 & a_{n-7}^1 & \cdots \\
n-2 & a_{n-2}^2 & a_{n-4}^2 & a_{n-6}^2 & \cdots & \\
n-3 & a_{n-3}^3 & a_{n-5}^3 & a_{n-7}^3 & \cdots & \\
\vdots & \vdots & \vdots & & & \\
3 & a_3^{n-3} & a_1^{n-3} & & & \\
2 & a_2^{n-2} & a_0^{n-2} & & & \\
1 & a_1^{n-1} & & & & \\
0 & a_0^n & & & &
\end{array}
$$

Table 3.3.1 - Routh-Hurwitz table

If the table can be completed (that means if the first column elements are all nontrivial, no roots are expected with null real part. Then we consider the sign of the first column elements. The number of sign variations between adjacent elements provides the number of roots with positive real part. In order to simplify computations, all the elements of a row can be multiplied per a strictly positive coefficient

Remark 3.3.7. A necessary condition required to obtain that all the roots have strictly negative real part, is that all the polynomial coefficients must have the same sign •

Remark 3.3.8. In case of a null element in the first column, the table cannot be completed. However if there are no other null elements in the row, the following alternative ways can be follow to complete the table:

a) substitute the null element with the infinitesimal quantity $\varepsilon > 0$, and keep on building the table and working on it. Assuming the infinitesimal quantity $\varepsilon < 0$ does not change the result;
b) multiply the original polynomial per a further polynomial with known nontrivial roots; then re-compute the coefficients and write down the table. This second approach is heuristic and does not guarantee to solve the problem. •

Example 3.3.9 Let $d(\lambda)$ be the characteristic polynomial of a linear system:

$$d(\lambda) = \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 2. \tag{3.3.3}$$

When building the Routh-Hurwitz table in order to investigate how many roots with positive real part come out, we have:

$$
\begin{array}{c|ccc}
5 & 1 & 1 & 1 \\
4 & 1 & 1 & 2 \\
3 & 0 & -1 &
\end{array}
$$

Since only the first element of the third row is null, we can complete the table by substituting $0 \mapsto \varepsilon > 0$:

$$
\begin{array}{c|ccc}
5 & 1 & 1 & 1 \\
4 & 1 & 1 & 2 \\
3 & \varepsilon & -1 & \\
2 & 1/\varepsilon & 2 & \\
1 & -1 & & \\
0 & 2 & &
\end{array}
$$

Note that we have taken into account the fact that ε is infinitesimal, therefore:

$$a_2^3 = \frac{1+\varepsilon}{\varepsilon} = \frac{1}{\varepsilon}, \qquad a_1^4 = -1 - 2\varepsilon^2 = -1. \tag{3.3.4}$$

By computing how many variations occur in the first column:

$$\begin{array}{ccccccccccc} 1 & & 1 & & \varepsilon & & \frac{1}{\varepsilon} & & -1 & & 2 \\ & P & & P & & P & & V & & V & \end{array}$$

we have 2 variations, that means 2 roots with positive real part, that means 2 eigenvalues with positive real part: the linear system is unstable. In case of choosing $\varepsilon < 0$, there would have been no change in the final result, indeed:

$$\begin{array}{ccccccccccc} 1 & & 1 & & \varepsilon & & \frac{1}{\varepsilon} & & -1 & & 2 \\ & P & & V & & P & & P & & V & \end{array}$$

If we use the second approach reported in Remark 3.3.8, we can multiply $d(\lambda)$ per $\lambda + 1$:

$$\tilde{d}(\lambda) = d(\lambda)(\lambda + 1) = \lambda^6 + 2\lambda^5 + 2\lambda^4 + 2\lambda^3 + 2\lambda^2 + 3\lambda + 2. \tag{3.3.5}$$

We are now able to build the Routh-Hurwitz table

$$\begin{array}{c|cccc} 6 & 1 & 2 & 2 & 2 \\ 5 & 2 & 2 & 3 & \\ 4 & 1 & 1/2 & 2 & \\ 3 & & & & \\ 2 & & & & \\ 1 & & & & \\ 0 & & & & \end{array} \quad \Longrightarrow \quad \begin{array}{c|cccc} 6 & 1 & 2 & 2 & 2 \\ 5 & 2 & 2 & 3 & \\ 4 & 2 & 1 & 4 & \\ 3 & 1 & -1 & & \\ 2 & 3 & 4 & & \\ 1 & -7/3 & & & \\ 0 & 4 & & & \end{array}$$

from which we obtain again 2 variations:

$$\begin{array}{ccccccccccccc} 1 & & 2 & & 2 & & 1 & & 3 & & -\frac{7}{3} & & 4 \\ & P & & P & & P & & P & & V & & V & \end{array}$$

•

Remark 3.3.10. When building the Routh-Hurwitz table for a polynomial like (3.3.2), in case *all* the elements of a row vanish, the following algorithm can be used:

i) the null-row is necessarily an odd row, and the polynomial can be factorized in: $p(\lambda) = p_1(\lambda)p_2(\lambda)$, with polynomials $p_1(\lambda)$ and $p_2(\lambda)$ described as follows;
ii) the analysis concerning the sign of the real part of the roots of $p_1(\lambda)$ is done by counting the sign variations of the first column of the incomplete table;
iii) polynomial $p_2(\lambda)$ has all even powers, and can be written by using the coefficients of the row immediately before the null-row; for instance, by assuming the following scheme:

$$\begin{array}{c|cccc} 6 & a_6^{n-6} & a_4^{n-6} & a_2^{n-6} & a_0^{n-6} \\ & & & & \\ 5 & 0 & 0 & 0 & \end{array}$$

we have:

$$p_2(\lambda) = a_6^{n-6}\lambda^6 + a_4^{n-6}\lambda^4 + a_2^{n-6}\lambda^2 + a_0^{n-6}; \tag{3.3.6}$$

iv) compute the derivative $\dfrac{dp_2}{d\lambda} = \tilde{p}(\lambda)$;
v) substitute in the Routh-Hurwitz table the null-row with the coefficients of $\tilde{p}(\lambda)$;
vi) complete the table and consider the sign variation in the first column referred only to the second part of the table;
vii) in this case a variation corresponds to a pair of roots with nontrivial real part and opposite sign;
viii) if the sum of the roots of $p_1(\lambda)$ and the pair of roots with nontrivial real part of $p_2(\lambda)$ is not n, the remaining roots of $d(\lambda)$ have null real part. •

Example 3.3.11. Let $d(\lambda)$ be the characteristic polynomial of a linear system:

$$d(\lambda) = \lambda^7 + 3\lambda^6 + 2\lambda^5 + 6\lambda^4 + 5\lambda^3 + 15\lambda^2 + 4\lambda + 12. \tag{3.3.7}$$

When building the Routh-Hurwitz table in order to investigate how many roots with positive real part come out, we have:

$$\begin{array}{c|cccc} 7 & 1 & 2 & 5 & 4 \\ 6 & 3 & 6 & 15 & 12 \\ 5 & & & & \end{array} \qquad \Longrightarrow \qquad \begin{array}{c|cccc} 7 & 1 & 2 & 5 & 4 \\ 6 & 1 & 2 & 5 & 4 \\ 5 & 0 & 0 & 0 & \end{array}$$

The 5-th row is made of all null elements. By applying the criterion suggested in Remark 3.3.10, we have no change in sign for the first part of the table, that means $d_1(\lambda)$ has just one root with negative real part. Then, we write $d_2(\lambda)$ as:

$$d_2(\lambda) = \lambda^6 + 2\lambda^4 + 5\lambda^2 + 4 \qquad \Longrightarrow \qquad d_2'(\lambda) = 6\lambda^5 + 8\lambda^3 + 10\lambda.$$

Finally, we substitute the 5-th null row with the coefficients of $d_2'(\lambda)$ and we complete the Routh-Hurwitz table:

$$\begin{array}{c|cccc} 7 & 1 & 2 & 5 & 4 \\ 6 & 1 & 2 & 5 & 4 \\ 5 & 3 & 4 & 5 & \\ 4 & 2/3 & 10/3 & 4 & \\ 3 & & & & \\ 2 & & & & \\ 1 & & & & \\ 0 & & & & \end{array} \quad \Longrightarrow \quad \begin{array}{c|cccc} 7 & 1 & 2 & 5 & 4 \\ 6 & 1 & 2 & 5 & 4 \\ 5 & 3 & 4 & 5 & \\ 4 & 1 & 5 & 6 & \\ 3 & -11 & -13 & & \\ 2 & 42/11 & 6 & & \\ 1 & & & & \\ 0 & & & & \end{array} \quad \Longrightarrow \quad \begin{array}{c|cccc} 7 & 1 & 2 & 5 & 4 \\ 6 & 1 & 2 & 5 & 4 \\ 5 & 3 & 4 & 5 & \\ 4 & 1 & 5 & 6 & \\ 3 & -11 & -13 & & \\ 2 & 7 & 11 & & \\ 1 & 30/7 & & & \\ 0 & 11 & & & \end{array}$$

When computing the number of variations in sign, starting from the 5-th row, we have:

$$\begin{array}{ccccccccccc} 3 & & 1 & & -11 & & 7 & & \frac{30}{7} & & 11 \\ & P & & V & & V & & P & & P & \end{array}$$

These 2 variations mean 2 pairs of roots with nontrivial real part and opposite sign, that means: 2 roots with positive real part and 2 roots of negative real part. Since the sum of the whole number of roots is 1 (coming from $d_1(\lambda)$) and 4 (coming from $d_2(\lambda)$) equal to 5, and we have a 7-th order polynomial, we still have to consider a couple of roots with null real part. Thus, the linear system is unstable. •

4 - QUALITATIVE BEHAVIOR OF SOLUTIONS

4.1 - THE PLANAR LINEAR CASE

In this section we investigate the qualitative behavior of the solutions of a linear system in the neighborhood of the origin. When dealing with second-order systems, a useful tool to this aim are the *vector field diagrams*, that are plots on the state space where at each point $\bar{x}$ of the plane is drawn a vector with the same direction of $f(\bar{x})$ and a length proportional to $\|f(\bar{x})\|$. Point after point, any trajectory related to the system under investigation follows the direction of the vector field. The set of all the trajectories is called the *phase portrait*.

Let us consider a second-order linear system:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in I\!R^2, \qquad A \in I\!R^{2\times 2}. \tag{4.1.1}$$

The goal of our investigation is to distinguish among different qualitative behaviors around the origin according to different sets of matrix A eigenvalues.

i) **Distinct, real, negative eigenvalues: the origin is a STABLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of two asymptotically stable aperiodic natural modes:

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in I\!R, \qquad \lambda_1 < \lambda_2 < 0. \tag{4.1.2}$$

In case of x_0 proportional to one of the two eigenvectors, e.g. u_1, the evolution reduces to:

$$x(t) = e^{\lambda_1 t} c_1 u_1, \qquad \lambda_1 < 0, \tag{4.1.3}$$

thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ approaches the origin along r_1 in an infinite time, with rate $|\lambda_1|$. Otherwise the evolution follows a more generic trajectory in the plane which again approaches the origin in an infinite time. In any case, no oscillations occur. Fig.4.1.1 reports the vector field diagram (in blue) when the origin is a stable node: the rightlines containing u_1 and u_2 are the x-axis and the other black line; red lines are some trajectories from the phase portrait, according to different initial states.

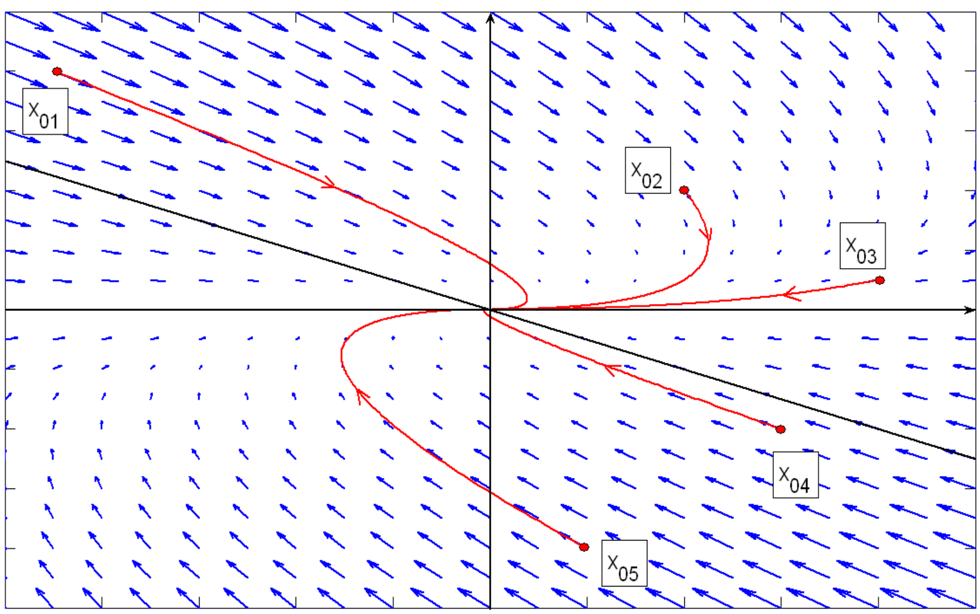

Fig. 4.1.1 - Vector field diagram of a stable node.

ii) **Distinct, real, positive eigenvalues: the origin is an UNSTABLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of two unstable aperiodic natural modes:

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in I\!R, \qquad 0 < \lambda_1 < \lambda_2. \tag{4.1.4}$$

In case of x_0 proportional to one of the two eigenvectors, e.g. u_1, the evolution reduces to:

$$x(t) = e^{\lambda_1 t} c_1 u_1, \qquad \lambda_1 > 0, \tag{4.1.5}$$

thus wherever x_0 is placed on the right line r_1 containing u_1, $x(t)$ escapes from the origin diverging to infinity along r_1 in an infinite time, with rate λ_1. Otherwise the evolution follows a more generic trajectory in the plane which again escapes from the origin diverging to infinity in an infinite time. No oscillations occur. Fig.4.1.2 reports the vector field diagram (in blue) when the origin is an unstable node: the rightlines containing u_1 and u_2 are the x-axis and the other black line; red lines are some trajectories from the phase portrait, according to different initial states.

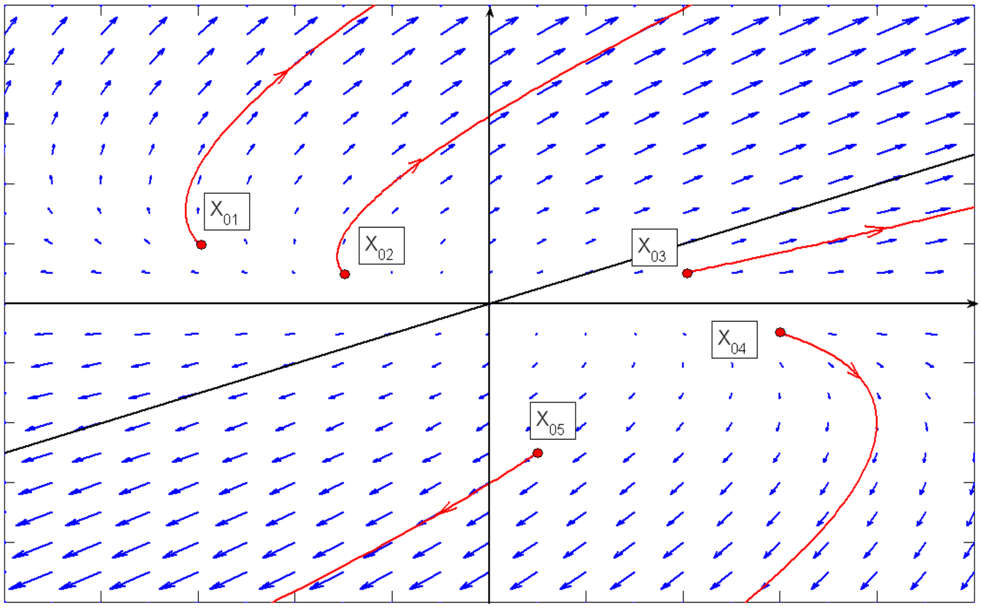

Fig. 4.1.2 - Vector field diagram of an unstable node.

iii) **Distinct, real, positive and negative eigenvalues: the origin is a SADDLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of an asymptotically stable + an unstable mode, both aperiodic:

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \qquad \lambda_1 < 0 < \lambda_2. \tag{4.1.6}$$

In case of x_0 proportional to the eigenvector u_1, related to the negative eigenvalue, the evolution reduces to the one in (4.1.3), thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ approaches the origin along r_1 in an infinite time, with rate $|\lambda|$; otherwise the evolution is composed by the linear combination of an increasing and a decreasing exponential, thus resulting in a generic trajectory always escaping from the origin and diverging to ∞ in an infinite time, asymptotically approaching the rightline r_2 containing u_2. The case of approaching the origin occurs only if the initial state lies on r_1. No oscillations occur. Fig.4.1.3 reports the vector field diagram (in blue) when the origin is a saddle node: the rightlines containing u_1 and u_2 are the x-axis and the other black line, respectively; red lines are some trajectories from the phase portrait, according to different initial states.

iv) **Complex eigenvalues, with negative real part: the origin is a STABLE FOCUS.** According to the spectral decomposition (3.2.14), the time evolution is an asymptotically stable pseudoperiodic mode thus, from (3.2.17):

$$x(t) = 2\rho e^{\alpha t}\Big(\cos(\beta t+\theta)u_\alpha - \sin(\beta t+\theta)u_\beta\Big), \qquad \lambda = \alpha \pm j\beta, \quad c = \rho e^{\pm j\theta}, \quad u = u_\alpha \pm j u_\beta, \quad \alpha < 0. \tag{4.1.7}$$

In this case, wherever x_0 is placed, the evolution is a spiral converging to the origin in an infinite time, with exponential decay rate $|\alpha|$. Fig.4.1.4 reports the vector field diagram (in blue) when the origin is a stable focus; red lines are some trajectories from the phase portrait, according to different initial states.

v) **Complex eigenvalues, with positive real part: the origin is an UNSTABLE FOCUS.** According to the spectral decomposition (3.2.14), the time evolution is an unstable pseudoperiodic mode thus, from (3.2.17):

$$x(t) = 2\rho e^{\alpha t}\Big(\cos(\beta t+\theta)u_\alpha - \sin(\beta t+\theta)u_\beta\Big), \qquad \lambda = \alpha \pm j\beta, \quad c = \rho e^{\pm j\theta}, \quad u = u_\alpha \pm j u_\beta, \quad \alpha > 0. \tag{4.1.8}$$

In this case, wherever x_0 is placed, the evolution is a spiral escaping from the origin, diverging to infinity in an infinite time, with exponential rate α. Fig.4.1.5 reports the vector field diagram (in blue) when the origin is an unstable focus; red lines are some trajectories from the phase portrait, according to different initial states.

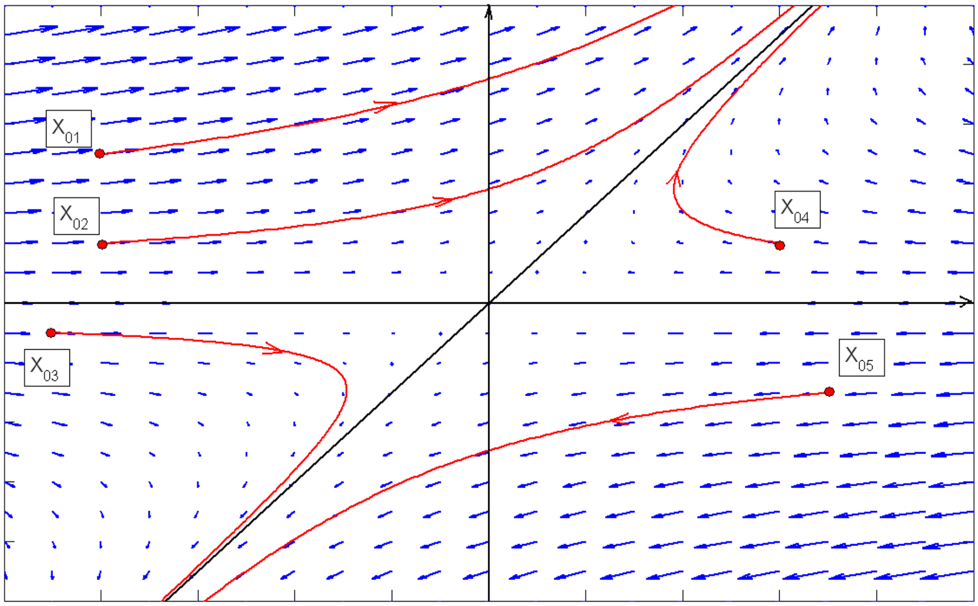

Fig. 4.1.3 - Vector field diagram of a saddle node.

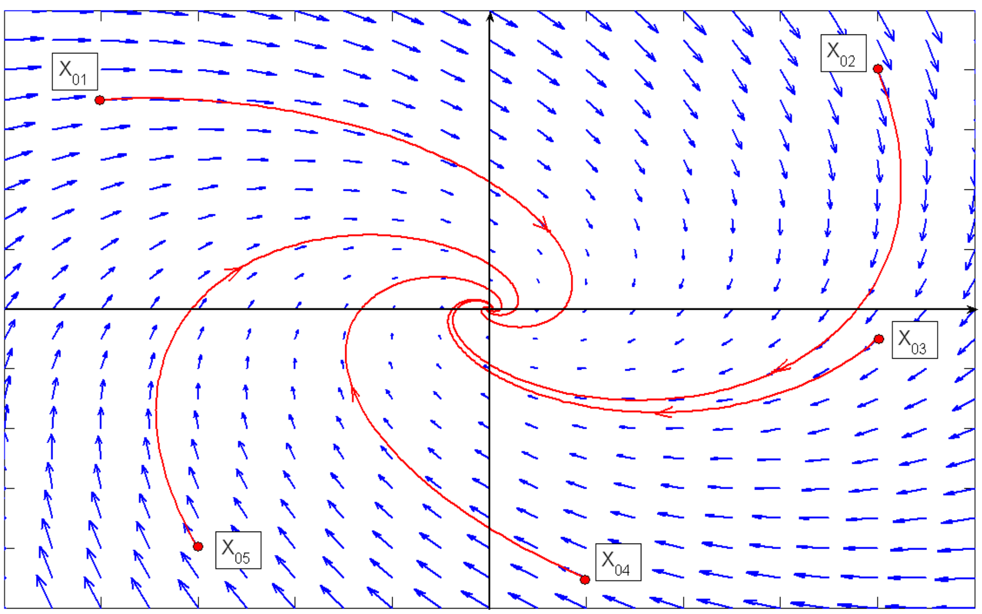

Fig. 4.1.4 - Vector field diagram of a stable focus.

vi) **Complex eigenvalues, with null real part: the origin is a CENTER.** According to the spectral decomposition (3.2.14), the time evolution is a stable periodic mode thus, from (3.2.16):

$$x(t) = 2\rho\Big(\cos(\beta t+\theta)u_\alpha - \sin(\beta t+\theta)u_\beta\Big), \qquad \lambda = \pm j\beta, \quad c = \rho e^{\pm j\theta}, \quad u = u_\alpha \pm j u_\beta. \tag{4.1.9}$$

In this case, wherever x_0 is placed, the evolution is a *closed orbit* with period equal to $2\pi/\beta$. It has to be stressed that these kind of periodic orbits are very sensitive to the model parameters, since the eigenvalue must have *exactly* null real part: any other value would produce stable/unstable foci. Moreover, the amplitude of these orbits depends on the initial condition: by increasing/decreasing the length ρ of the initial condition vector, the amplitude of the orbit increases/decreases. Fig.4.1.6 reports the vector field diagram (in blue) when the origin is center; red lines are some trajectories from the phase portrait, according to different initial states.

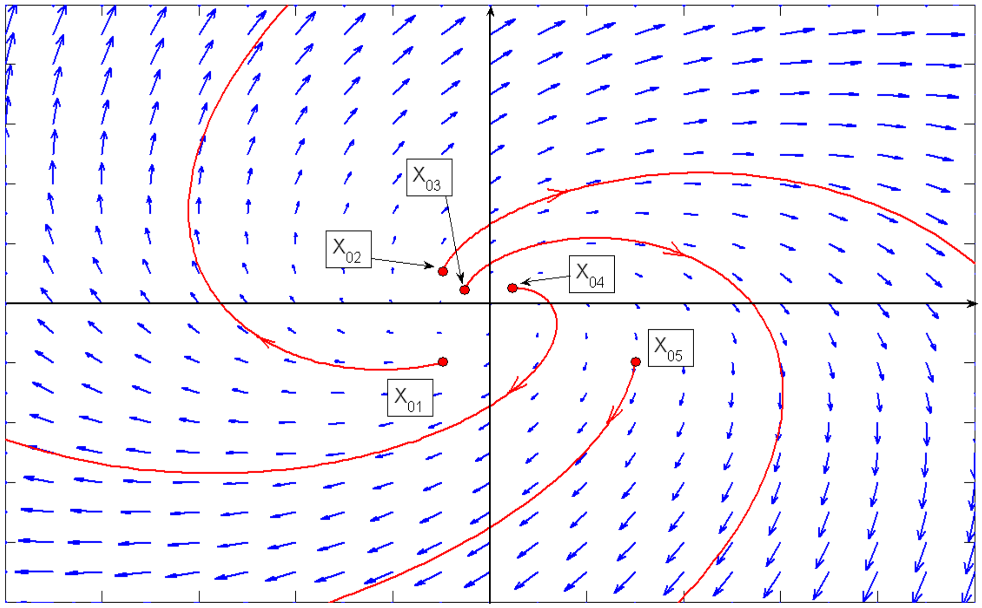

Fig. 4.1.5 - Vector field diagram of an unstable focus.

Other possibilities may occur, but they produce results that can be represented by the previous cases.

vii) **One multiple, real, negative eigenvalue: the origin behaves like a STABLE NODE.** According to the spectral decomposition (3.2.18), concerning multiple eigenvalues, we can have the following two cases:

vii-a) polynomials $p_i(t)$, $i = 1, 2$, in (3.2.18) have both degree 0:

$$x(t) = e^{\lambda_1 t}\big(c_1 u_1 + c_2 u_2\big), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 < 0, \tag{4.1.10}$$

with both u_1, u_2 non-generalized eigenvectors. The position of the initial state in the state plane ($x_0 = c_1 u_1 + c_2 u_2$) gives us the direction r according to which the evolution converges to the origin in an exponential fashion with rate $|\lambda_1|$: the trajectory is a segment on r, since the direction does not change with time t. Fig.4.1.7 reports the vector field diagram (in blue) when the origin behaves like a stable node with a multiple eigenvalue but no generalized eigenvectors; red lines are some trajectories from the phase portrait, according to different initial states.

vii-b) one of the two polynomials in (3.2.18) has degree 1:

$$x(t) = e^{\lambda_1 t}\big((c_1 + t c_2) u_1 + c_2 u_2\big), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 < 0, \tag{4.1.11}$$

where c_1 and c_2 are the coordinates of the initial state with respect to u_1 (non-generalized eigenvector) and u_2 (generalized eigenvector), respectively. In case of x_0 proportional to u_1, the evolution reduces to the one in (4.1.3), thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ approaches the origin along r_1 in an infinite time, with rate $|\lambda_1|$. Otherwise, there are no other cases with no change in the direction since, even if x_0 is proportional to u_2, equation (4.1.10) becomes:

$$x(t) = e^{\lambda_1 t} c_2 \big(t u_1 + u_2\big), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 < 0. \tag{4.1.12}$$

Therefore, if x_0 does not belong to r_1 we have a generic curve converging to the origin with no oscillations. In any case, all the curves converge to the origin with a slope asymptotically converging to the one of r_1 since, for great values of time t, it is $(c_1+tc_2)u_1+c_2u_2 \simeq tc_2u_1$. Fig.4.1.8 reports the vector field diagram (in blue) when the origin behaves like a stable node with a multiple eigenvalue and generalized eigenvectors; the black line gives the direction r_1 of the non-generalized eigenvector; red lines are some trajectories from the phase portrait, according to different initial states.

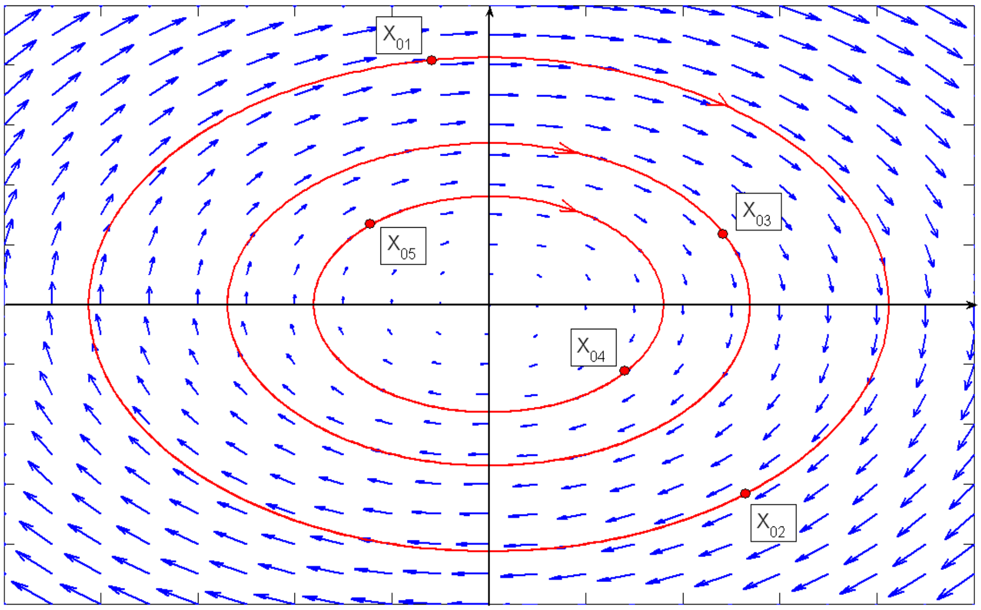

Fig. 4.1.6 - Vector field diagram of center.

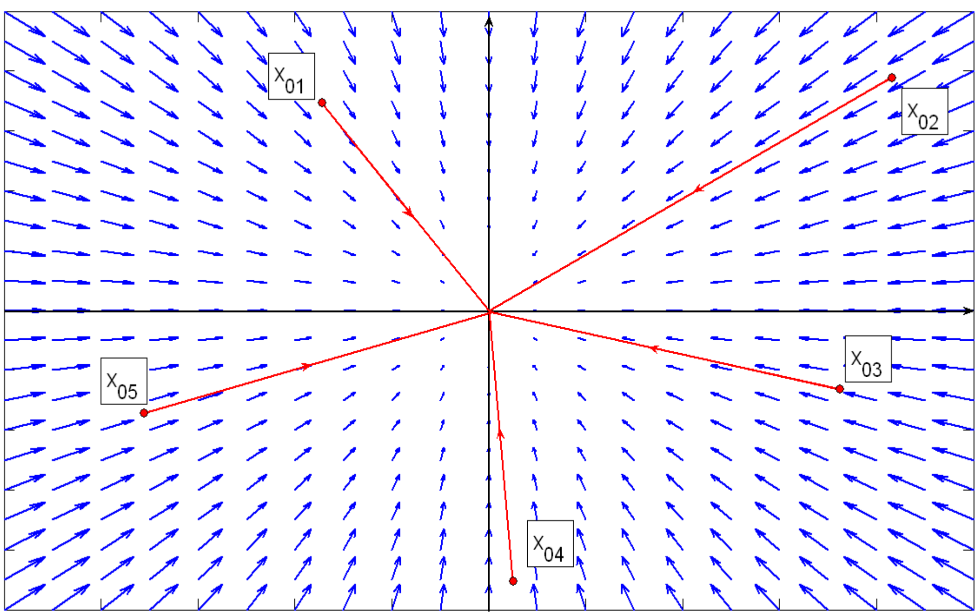

Fig. 4.1.7 - Vector field diagram for a multiple real negative eigenvalue without generalized eigenvectors.

viii) **One multiple, real, positive eigenvalue: the origin behaves like an UNSTABLE NODE.** According to the spectral decomposition (3.2.18), concerning multiple eigenvalues, we can have the following two cases:

viii-a) polynomials $p_i(t)$, $i = 1, 2$, in (3.2.18) have both degree 0:

$$x(t) = e^{\lambda_1 t}(c_1u_1 + c_2u_2), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 > 0, \tag{4.1.13}$$

with both u_1, u_2 non-generalized eigenvectors. The position of the initial state in the state plane

($x_0 = c_1u_1 + c_2u_2$) gives us the direction r according to which the evolution escapes from the origin to infinity in an exponential fashion with rate λ_1: the trajectory is a halfline of r, since the direction does not change with time t. Fig.4.1.9 reports the vector field diagram (in blue) when the origin behaves like an unstable node with a multiple eigenvalue but no generalized eigenvectors; red lines are some trajectories from the phase portrait, according to different initial states.

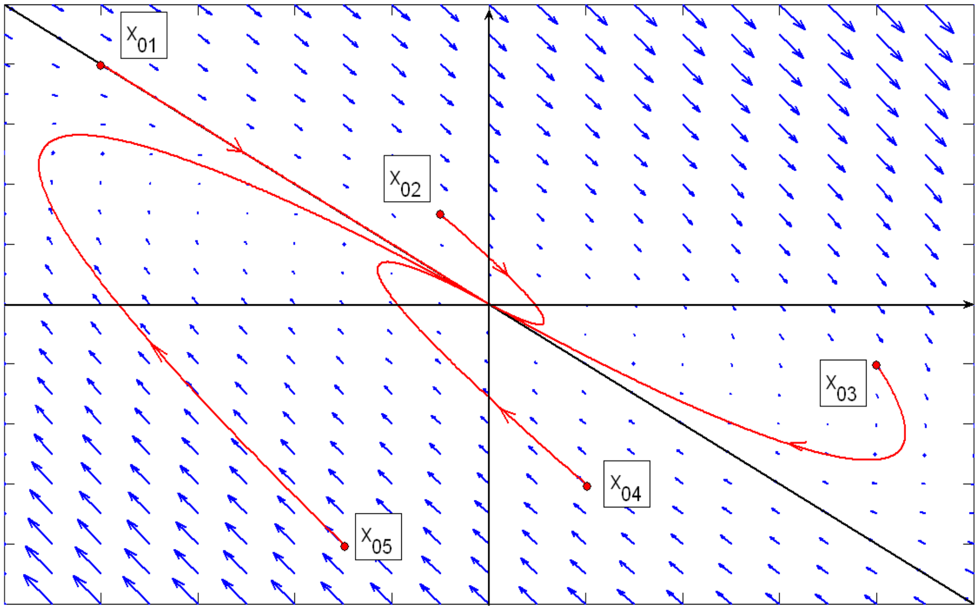

Fig. 4.1.8 - Vector field diagram for a multiple real negative eigenvalue with generalized eigenvectors.

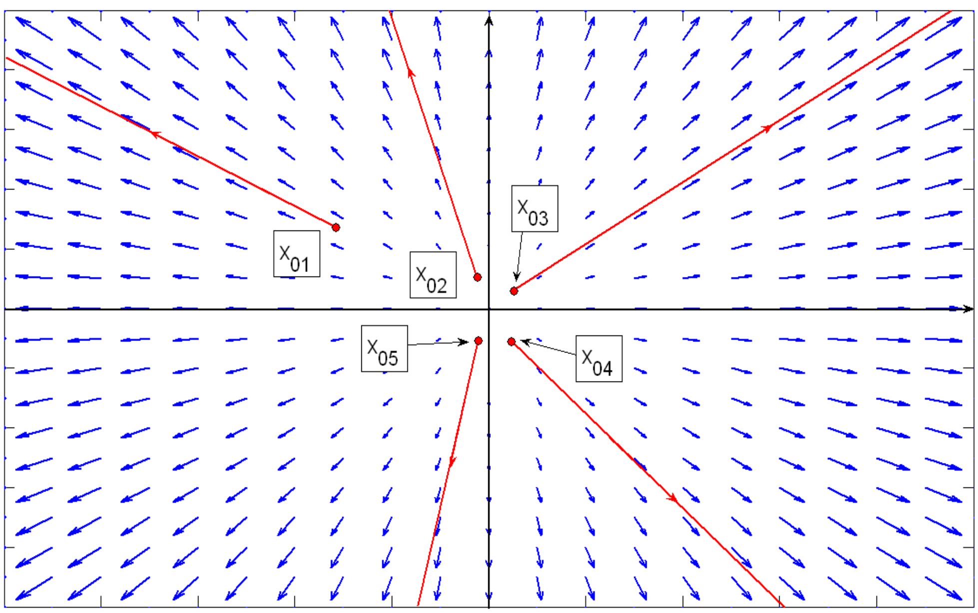

Fig. 4.1.9 - Vector field diagram for a multiple real positive eigenvalue without generalized eigenvectors.

viii-b) one of the two polynomials in (3.2.18) has degree 1:

$$x(t) = e^{\lambda_1 t}\big((c_1 + tc_2)u_1 + c_2u_2\big), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 > 0, \tag{4.1.14}$$

where c_1 and c_2 are the coordinates of the initial state with respect to u_1 (non-generalized eigenvector) and u_2 (generalized eigenvector), respectively. In case of x_0 proportional to u_1, the evolution reduces to the one in (4.1.5), thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ escapes from the origin to infinity along r_1 in an infinite time, with rate λ_1. Otherwise, there are no other cases with no change in the direction since, even if x_0 is proportional to u_2, equation (4.1.13) becomes:

$$x(t) = e^{\lambda_1 t} c_2 (t u_1 + u_2), \qquad \lambda_1 \in I\!R, \qquad \lambda_1 > 0. \tag{4.1.15}$$

Therefore, if x_0 does not belong to r_1 we have a generic curve escaping from the origin to infinity with no oscillations. In any case, all the diverging curves have a slope converging to the one of r_1 since, for great values of time t, it is $(c_1 + tc_2)u_1 + c_2 u_2 \simeq tc_2 u_1$. Fig.4.1.10 reports the vector field diagram (in blue) when the origin behaves like a stable node with a multiple eigenvalue and generalized eigenvectors; the black line gives the direction r_1 of the non-generalized eigenvector; red lines are some trajectories from the phase portrait, according to different initial states.

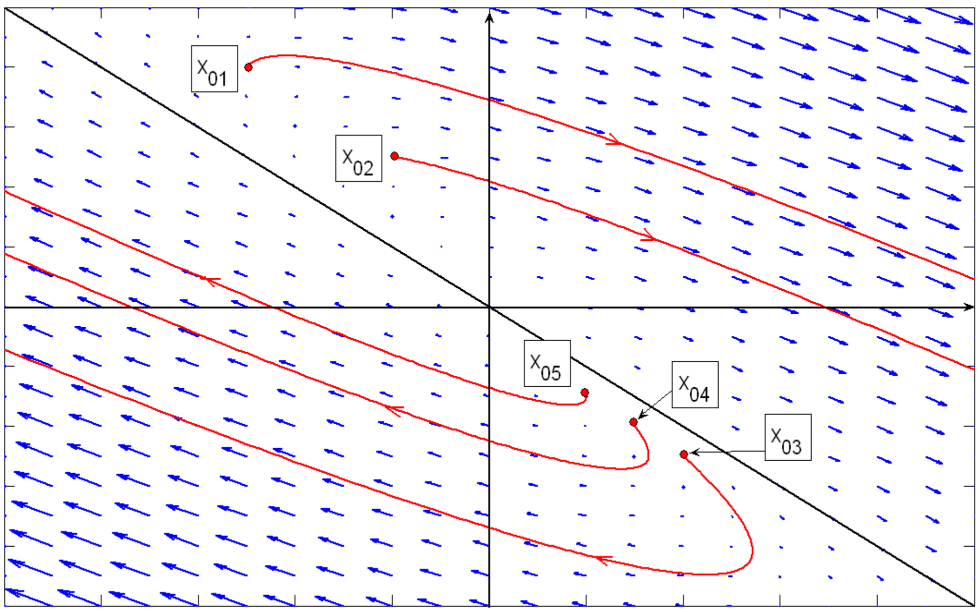

Fig. 4.1.10 - Vector field diagram for a multiple real positive eigenvalue with generalized eigenvectors.

All previous cases deal with the origin as the only equilibrium point of the linear system. Below follow three other cases, known as *degenerate* cases, involving a singular matrix A: the origin is no more the only equilibrium point.

ix) **Distinct, real, eigenvalues; one null, the other negative: DEGENERATE STABLE CASE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of a stable (null eigenvalue) + an asymptotically stable aperiodic mode:

$$x(t) = c_1 u_1 + e^{\lambda t} c_2 u_2, \qquad \lambda \in I\!R, \qquad \lambda < 0. \tag{4.1.16}$$

In case of x_0 proportional to u_1, the eigenvector related to the null eigenvalue, the evolution reduces to: $x(t) = c_1 u_1$, thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ does not move with time and the trajectory is trivially the point $c_1 u_1$. Indeed, the points on r_1 are all equilibrium points. Otherwise the evolution will follow a straight line converging to the line r_1, according to a direction which is the same of u_2. Fig.4.1.11 reports the vector field diagram (in blue) where u_1 belongs to the x-axis, and u_2 is the other black line; red lines are some trajectories from the phase portrait, according to different initial states.

x) **Distinct, real, eigenvalues; one null, the other positive: DEGENERATE UNSTABLE CASE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of a stable (null eigenvalue) + an unstable aperiodic mode:

$$x(t) = c_1 u_1 + e^{\lambda t} c_2 u_2, \qquad \lambda \in I\!R, \qquad \lambda > 0. \tag{4.1.17}$$

In case of x_0 proportional to u_1, the eigenvector related to the null eigenvalue, the evolution reduces to: $x(t) = c_1 u_1$, thus wherever x_0 is placed on the rightline r_1 containing u_1, $x(t)$ does not move with time and the trajectory is trivially the point $c_1 u_1$. Indeed, the points on r_1 are all equilibrium points. Otherwise the evolution will follow a straight line escaping from the line r_1 and diverging to infinity, according to a direction which is the same of u_2. Fig.4.1.12 reports the vector field diagram (in blue) where u_1 belongs to the x-axis, and u_2 is the other black line; red lines are some trajectories from the phase portrait, according to different initial states.

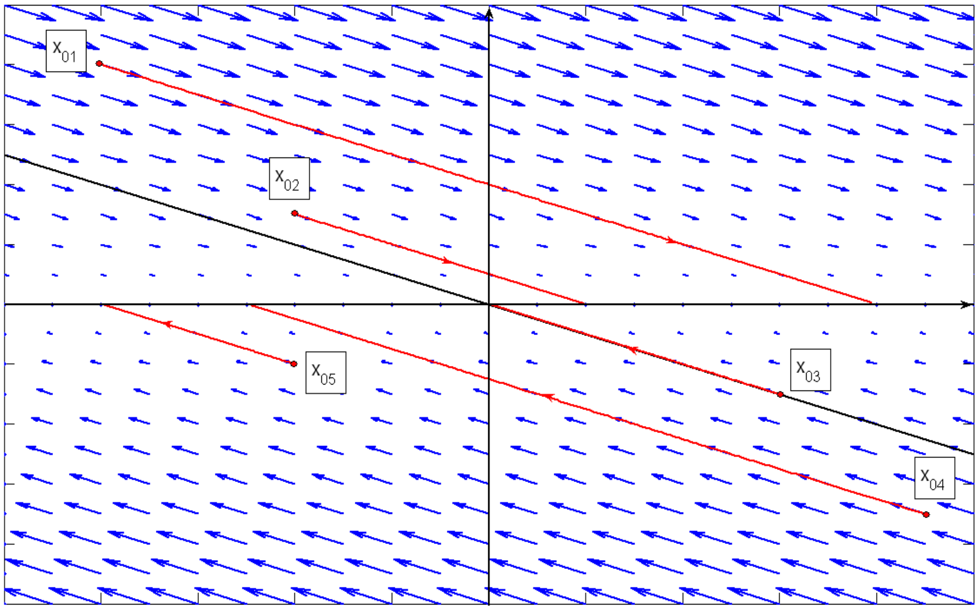

Fig. 4.1.11 - Vector field diagram for distinct real eigenvalues: one null, the other negative.

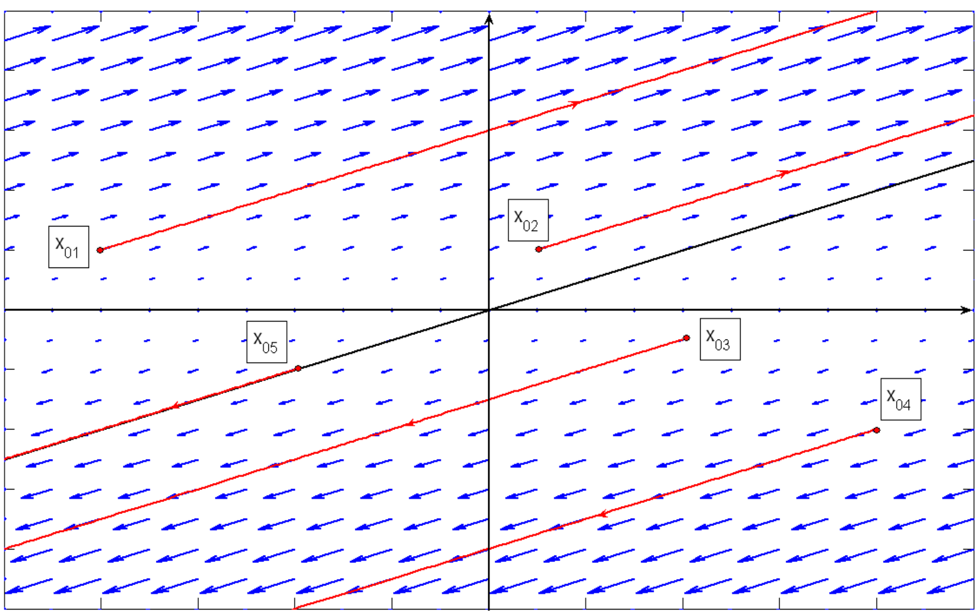

Fig. 4.1.12 - Vector field diagram for distinct real eigenvalues: one null, the other positive.

xi) **One multiple, null eigenvalue.** According to the spectral decomposition (3.2.18), concerning multiple eigenvalues, we can have the following two cases:

xi-a) polynomials $p_i(t)$, $i = 1, 2$, in (3.2.18) have both degree 0:

$$x(t) = c_1 u_1 + c_2 u_2, \tag{4.1.18}$$

with both u_1, u_2 non-generalized eigenvectors. As it clearly appears from (4.1.18) there is no motion wherever is the initial state x_0: indeed, any point is an equilibrium point and all possible trajectories are trivially points in the state space. This is a *degenerate stable case.*

xi-b) one of the two polynomials in (3.2.18) has degree 1:

$$x(t) = (c_1 + tc_2)u_1 + c_2u_2, \tag{4.1.19}$$

where c_1 and c_2 are the coordinates of the initial state with respect to u_1 (non-generalized eigenvector) and u_2 (generalized eigenvector), respectively. In case of x_0 proportional to u_1, the evolution reduces to:

$$x(t) = c_1u_1, \tag{4.1.20}$$

thus wherever x_0 is placed on the rightline r_1 containing u_1, there is no motion: indeed any point in r_1 is an equilibrium point. Otherwise, if x_0 does not belong to r_1, we have a generic curve escaping from the origin to infinity, with no oscillations, along a rightline parallel to r_1. The vector field diagram is the same of Fig.4.1.12; the black line gives the direction r_1 of the non-generalized eigenvector; red lines are some trajectories from the phase portrait, according to different initial states.

4.2 - THE MULTIDIMENSIONAL LINEAR CASE

Let us consider a generic n-dimensional linear system like the one defined in (3.1.1). When dimension n is greater than 2, vector field diagrams and phase portraits are quite more difficult to represent (they could be generalized for $n = 3$ on the Euclidean space) and much more hard to interpret. Moreover also the range of possibilities increases. For these reasons in this Section only most important n-dimensional generalizations will be considered.

i) **All distinct, real, negative eigenvalues: the origin is a STABLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of n asymptotically stable aperiodic natural modes. In case of x_0 proportional to a subset $\{u_1, \ldots, u_m\}$ of the eigenvectors (let us assume without loss of generality the first $m < n$), the evolution $x(t)$ converges to the origin in an infinite time, evolving on the subspace span$\{u_1, \ldots, u_m\}$. Otherwise the evolution follows a more generic trajectory in the state space which again approaches the origin in an infinite time. In any case, no oscillations occur.

ii) **All distinct, real, positive eigenvalues: the origin is an UNSTABLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of n unstable aperiodic natural modes. In case of x_0 proportional to a subset $\{u_1, \ldots, u_m\}$ of the eigenvectors (let us assume without loss of generality the first $m < n$), the evolution $x(t)$ escapes from the origin to infinity in an infinite time, evolving on the subspace span$\{u_1, \ldots, u_m\}$. Otherwise the evolution follows a more generic trajectory in the state space which again escapes from the origin to infinity in an infinite time. In any case, no oscillations occur.

iii) **All distinct, real, eigenvalues, some positive, some negative: the origin is a SADDLE NODE.** According to the spectral decomposition (3.2.14), the time evolution is the sum of some (let us say $n_1 < n$) asymptotically stable aperiodic and some other ($n_2 = n - n_1$) unstable aperiodic natural modes. In case of x_0 proportional to a subset of the eigenvectors related to the asymptotically stable modes, the evolution $x(t)$ converges to the origin in an infinite time, evolving on the subspace generated by those eigenvectors; on the other hand, in case of x_0 proportional to a subset of the eigenvectors related to the unstable modes, the evolution $x(t)$ escapes from the origin to infinity in an infinite time, evolving on the subspace generated by those eigenvectors. In the general framework, the evolution $x(t)$ escapes from the origin to infinity in an infinite time, with the evolution asymptotically converging to the subspace generated by the eigenvectors related to the unstable modes. In any case, no oscillations occur.

iv) **All distinct eigenvalues, some complex, all with negative real part: the origin is a STABLE FOCUS.** According to the spectral decomposition (3.2.14), the time evolution is the sum of asymptotically stable aperiodic and pseudoperiodic modes. In case of x_0 proportional to a subset of the eigenvectors related to the aperiodic modes, the evolution $x(t)$ converges to the origin in an infinite time, evolving on the subspace generated by these eigenvectors with no oscillations (actually behaving as if the origin were a stable node); otherwise in the general case, the pseudo-periodic modes make it so that the evolution is a spiral converging to the origin in an infinite time, with oscillations.

v) **All distinct eigenvalues, some complex, all with positive real part: the origin is an UNSTABLE FOCUS.** According to the spectral decomposition (3.2.14), the time evolution is the sum of unstable aperiodic and pseudoperiodic modes. In case of x_0 proportional to a subset of the eigenvectors related to the aperiodic modes, the evolution $x(t)$ escapes from the origin to infinity in an infinite time, evolving on the subspace generated by these eigenvectors with no oscillations (actually behaving as if the origin were an unstable node); otherwise in the general case, the pseudoperiodic modes make it so that the evolution is a spiral escaping from the origin to infinity in an infinite time, with oscillations.

vi) **All distinct eigenvalues, some complex with pure imaginary eigenvalues, the others negative real: the origin is a CENTER.** According to the spectral decomposition (3.2.14), the time evolution is the sum of asymptotically stable aperiodic and stable periodic modes. In case of x_0 proportional to a subset of the eigenvectors related to the aperiodic modes, the evolution $x(t)$ converges to the origin in an infinite time, evolving on the subspace generated by these eigenvectors with no oscillations (actually behaving as if the origin were a stable node); otherwise in the general case, the stable periodic modes make it so that the evolution converges to a closed orbit in the subspace generated by the eigenvectors related to the periodic modes.

4.3 - THE GENERAL NONLINEAR CASE

All the previous qualitative behaviors around the equilibrium points of linear systems have the common denominator of being *global*. For instance, consider the case of a saddle node: if the initial condition belongs to the subspace generated by only eigenvectors related to the asymptotically stable eigenvalues, the time evolution will definitely converge to the origin, no matter how far from the origin it is. These properties no longer belong to the qualitative behavior around the equilibrium points of nonlinear system which, therefore, is just a *local* qualitative behavior.

Let us consider a time-invariant, nonlinear system as the one defined in (2.1.1) with x_e one of its equilibrium points (recall that nonlinear systems may have many equilibrium points). According to the Taylor series expansion around x_e, system (2.1.1) can be written as:

$$\dot{x}(t) = f(x_e) + J(x_e)\big(x(t) - x_e\big) + h\big(x(t) - x_e\big), \tag{4.3.1}$$

with:

$$J(x) = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \tag{4.3.2}$$

the Jacobian matrix of function $f(\cdot)$ and $h(\cdot) : I\!R^n \mapsto I\!R^n$ the residual of the first-order Taylor expansion, such that:

$$\lim_{\|z\| \mapsto 0} \frac{\big\|h\big(\|z\|\big)\big\|}{\|z\|} = 0. \tag{4.3.3}$$

Remark 4.3.1. Define the displacement $z(t) = x(t) - x_e$. Note that its dynamics is given by:

$$\dot{z}(t) = J(x_e)z(t) + h\big(z(t)\big). \tag{4.3.4}$$

Thus, the qualitative behavior of $x(t)$ around the equilibrium point x_e is the same as the one of $z(t)$ around the origin. Moreover, if $J(x_e)$ has no null eigenvalues, if $z(t)$ is close enough to the origin, then the residual $h(\cdot)$ can be neglected with respect to the linear part (thanks to the limit (4.3.3)); on the other hand, if $J(x_e)$ has a null eigenvalue, then there exists at least one direction according to which $\dot{z}(t) = h\big(z(t)\big)$, that means $h(\cdot)$ cannot be neglected. •

According to Remark 4.3.1, if the origin of the linear approximation $\dot{z}(t) = J(x_e)z(t)$ is a stable-unstable-saddle node or a stable-unstable focus, then the equilibrium point x_e exhibits the qualitative *local* behavior of a stable-unstable-saddle node or of a stable-unstable focus, since there exists a neighborhood of the equilibrium point small enough to allow the linear approximation very well resemble the original nonlinear evolution.

Nonlinear systems can provide *periodic orbits* whose characteristics are different from the ones of a linear *center*. In the linear case they are very sensitive to the system parameters, and strongly depend of the initial condition: we can have a continuum of periodic orbits by varying the initial conditions. On the other hand, nonlinear ODE can produce isolated periodic orbits (called **limit cycles**), which the system evolution definitely approaches as time t goes to $+\infty$. Differently from the linear case they do not depend on the initial condition.

Remark 4.3.2. A limit cycle γ can be:
- *stable*, if for any nontrivial (i.e. with nontrivial measure) region $\mathcal{D}_\varepsilon$ containing the limit cycle γ, there exists a nontrivial inner region $\mathcal{D}_\delta \subseteq \mathcal{D}_\varepsilon$ containing γ such that, for any initial point $x_0 \in \mathcal{D}_\delta$ the evolution $x(t)$ is constrained in $\mathcal{D}_\varepsilon$ and converges to the limit cycle γ. In Fig.4.3.1, the vector field diagram of a stable limit cycle in the planar case is shown;
- *unstable* if, no matter how close (inner or outer) to the limit cycle the initial condition is, the evolution definitely escapes from γ. In Fig.4.3.2, the vector field diagram of an unstable limit cycle is shown, with a locally asymptotically stable inner equilibrium point. •

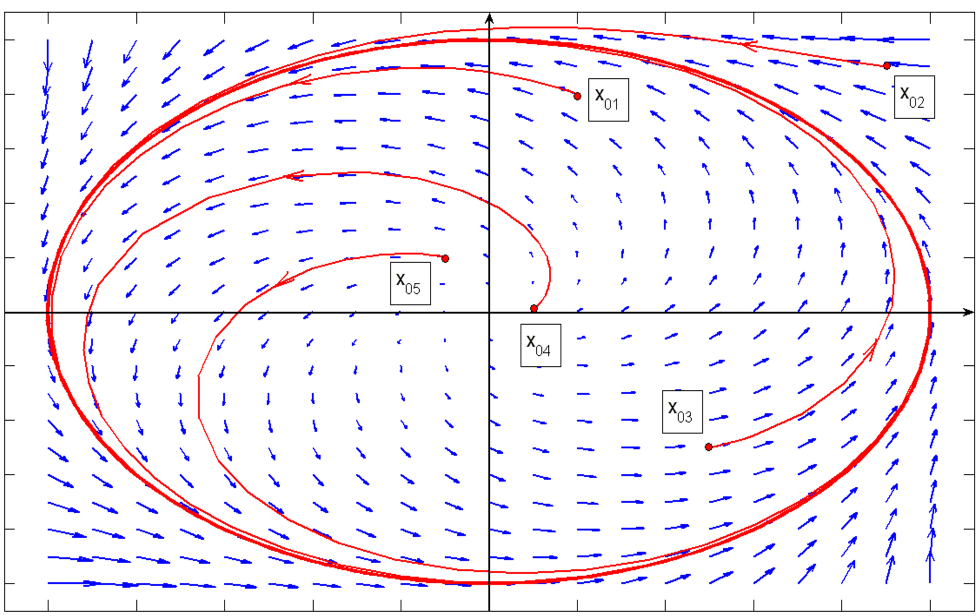

Fig. 4.3.1 - Vector field diagram for a stable limit cycle.

In case of time-invariant, second-order systems, the following criterion holds true:

Theorem 4.3.3 (Poincaré-Bendixon Criterion). *Consider a second-order time-invariant nonlinear system and D a closed bounded subset such that:*

i) every trajectory starting from a point in D remains in D;

ii) D does not contain equilibrium points, or contains only one unstable node or only one unstable focus. Then, D contains a limit cycle.

Remark 4.3.4. The Poincaré-Bendixon Criterion may be restated saying that, for time-invariant, second order systems, whenever we know that the trajectory is bounded in a closed subset, then there can be convergence to an inner equilibrium point or to an inner limit cycle. •

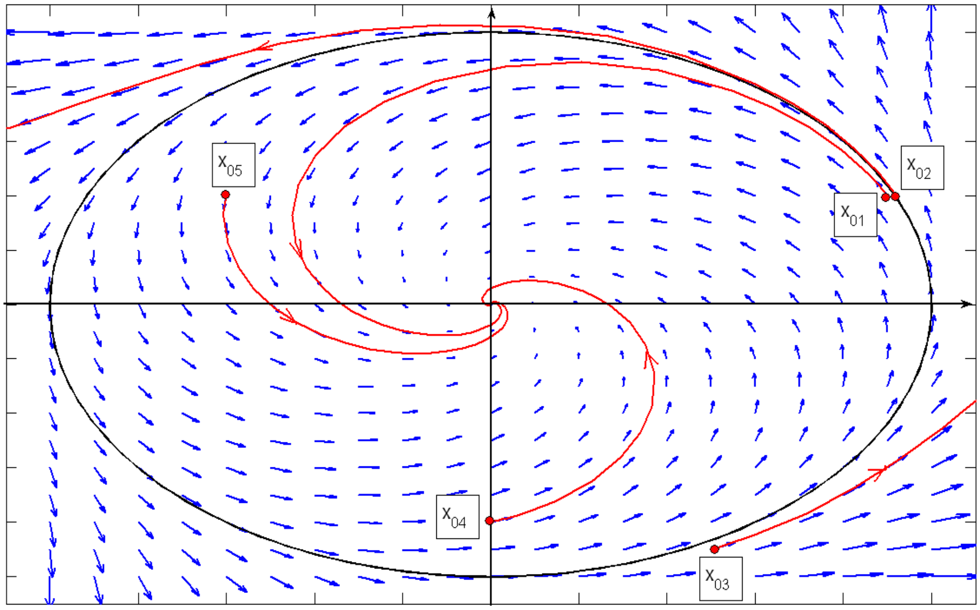

Fig. 4.3.2 - Vector field diagram for an unstable limit cycle.

In case of nonlinear systems, a further qualitative behavior can occur, known as **chaos**. It occurs when, regardless how close a couple of initial conditions are chosen, the two evolutions will definitely become completely different as time t increases. In Fig.4.3.3, a second-order, time-varying system is considered: it clearly appears that after a transient during which the two evolutions (one in blue, the other in red) are indistinguishable because of the choice of two initial states very close one each other, they follow two distinct ways of evolution. Chaotic systems are extremely sensitive to the initial conditions: a very small perturbation definitely produces a significative change in the evolution.

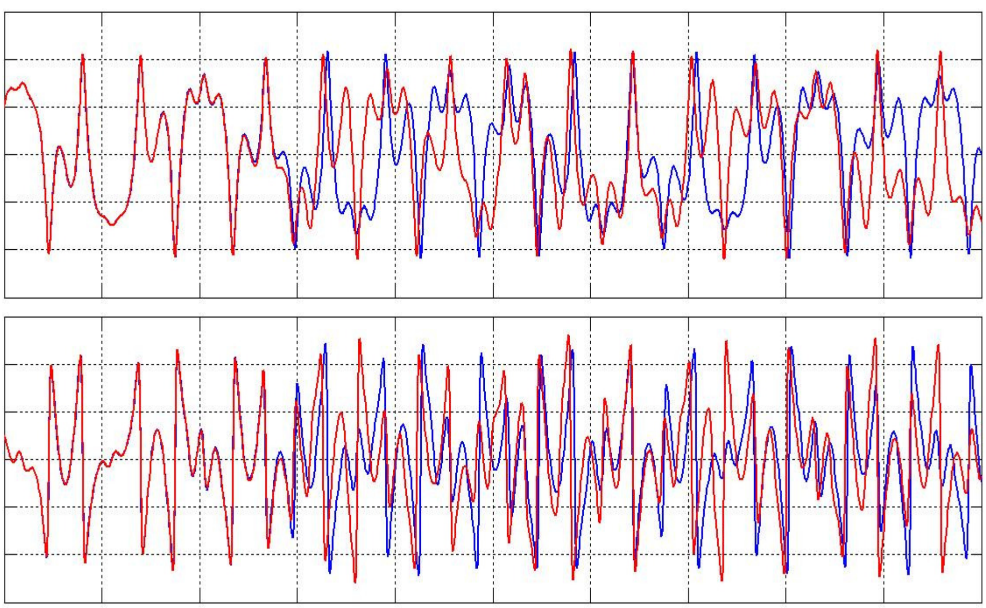

Fig. 4.3.3 - Chaotic time evolutions for the components of a second-order, time-varying nonlinear system.

Although a chaotic system does not provide limit cycles, the evolution in the state space is as well bounded in a region known as the *attractor*. Therefore, according to the Poincaré-Bendixon criterion a chaotic behavior cannot occur for *time-invariant* second-order systems; on the other hand, for third-order (or higher-order) systems, curves may be slanted and a state evolution can persists in a bounded region without ever allowing intersections. In Fig.4.3.4 the chaotic attractor related to the chaotic evolutions of Fig.4.3.3 is reported.

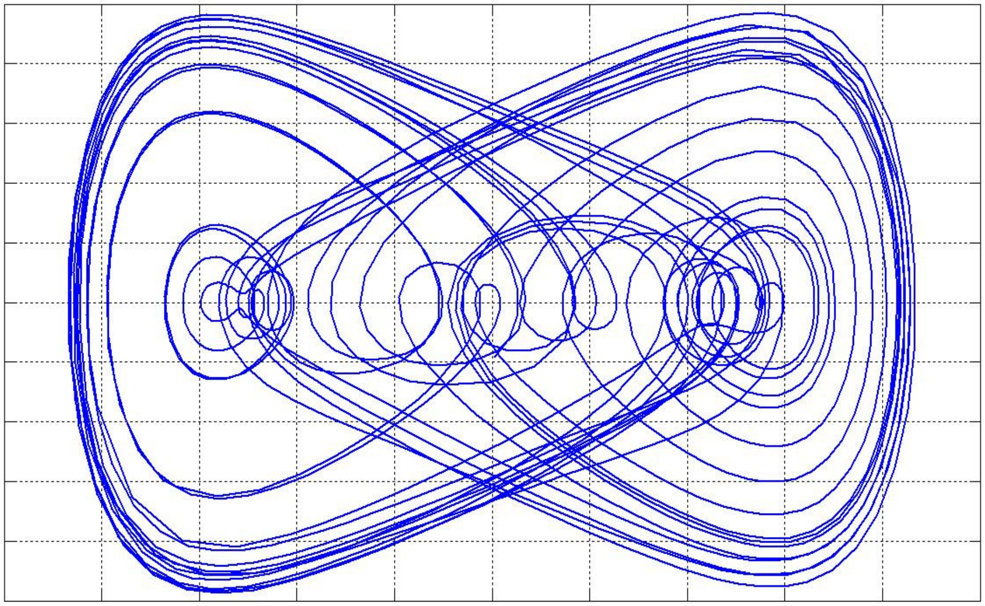

Fig. 4.3.4 - A chaotic attractor for a second-order, time-varying nonlinear system.

4.4 - BIFURCATION

Bifurcation theory takes into account what happens to the qualitative behavior around an equilibrium point when one or more parameters vary around their nominal values. To this aim the equilibrium points can be divided in *hyperbolic* and *non-hyperbolic*. The formers refer to the case of eigenvalues (of the Jacobian matrix) with non trivial real part. These equilibrium points show *structural stability* since small changes in the model parameters (which produce small changes in the placement of the eigenvalues on the complex plane) do not change the feature of the equilibrium point (an asymptotically stable node/focus is still an asymptotically stable node/focus according to small variation of a system parameter as well as unstable nodes/foci and saddle nodes). On the other hand, non-hyperbolic equilibrium points refer to the case of trivial (null) real part. In this case a small variation in a system parameter may well change the feature of the qualitative behavior: for instance, a center in a linear system may change into an asymptotically stable focus or into an unstable focus.

In this section, the case of only one varying parameter will be considered. In this framework, a **bifurcation** is a change in the qualitative behavior occurring when a parameter reaches a given value (the **bifurcation value**). They are generally described by **bifurcation diagrams** which in their simplest representation are planar diagrams referring the position of the equilibria versus the bifurcation parameter. Below follow classical bifurcation cases.

a) **SADDLE-NODE BIFURCATION.** It occurs when, by crossing the bifurcation value, a pair of equilibrium points (usually an asymptotically stable and an unstable/saddle node) converge to the same point and then both vanish. For instance, consider the system:

$$\dot{x}(t) = \mu - x^2(t). \tag{4.4.1}$$

It admits a couple of equilibrium points when $\mu > 0$ in $x_1 = \sqrt{\mu}$ and $x_2 = -\sqrt{\mu}$; x_1 is a stable node while x_2 is an unstable node. Otherwise, no equilibrium points exist. $\mu = 0$ is the bifurcation value. In Fig.4.4.1 the saddle-node bifurcation diagram is reported: the continuous line refers to the asymptotically stable equilibrium point, while the dotted line refers to the unstable equilibrium point.

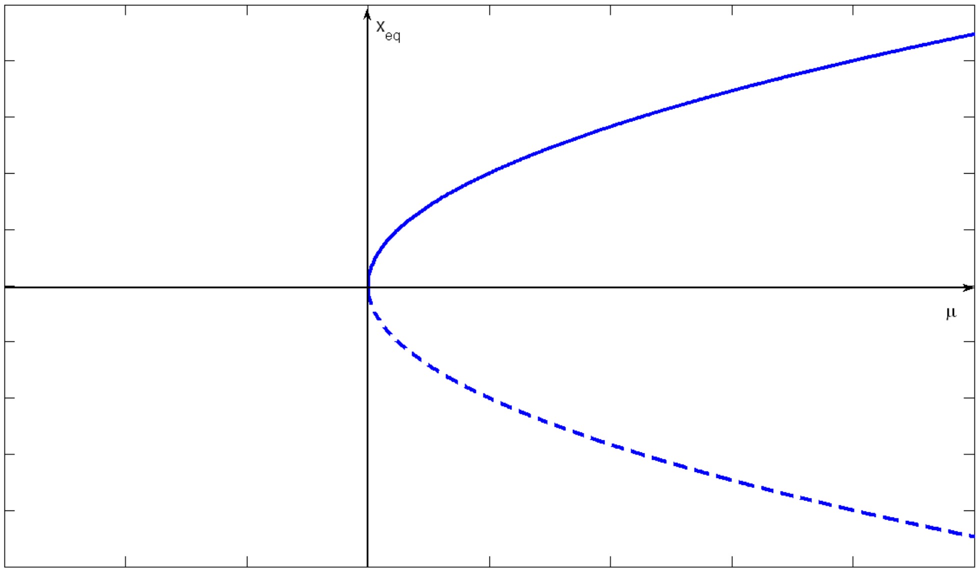

Fig. 4.4.1 - Saddle-node bifurcation diagram.

This kind of bifurcation is known as *dangerous* (or *hard*), since it introduces a strong change of behavior before and after the crossing of the bifurcation value. Imagine μ assumes a positive small value: even if small, its positivity provides that all the trajectories starting from $x_0 > -\sqrt{\mu}$ converge to $\sqrt{\mu}$. On the contrary, if μ assumes a negative small value, wherever is placed the initial state x_0, the trajectory will definitely diverge to $-\infty$. In this case, a small perturbation able to change the sign of μ will make a dramatic change in the qualitative behavior (see the vector field diagram of Fig.4.4.2).

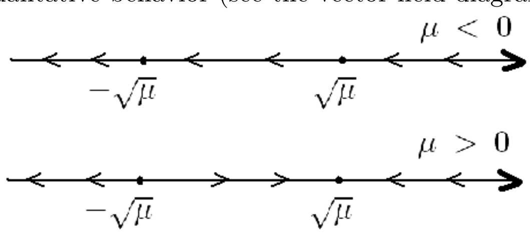

Fig. 4.4.2 - Vector field diagram for system (4.4.1).

b) **TRANSCRITICAL BIFURCATION.** It occurs when, by crossing the bifurcation value, a pair of equilibrium points with opposite qualitative behavior change their stability properties (e.g. the asymptotically stable changes into unstable and viceversa). No change occurs in the number of the equilibrium points. For instance, consider the system:

$$\dot{x}(t) = \mu x(t) - x^2(t). \tag{4.4.2}$$

For any value of the bifurcation parameter μ, the system admits two distinct equilibrium points: $x_1 = 0$ and $x_2 = \mu$. For $\mu < 0$ it is x_1 asymptotically stable and x_2 unstable, while for $\mu > 0$ it is x_1 unstable and x_2 asymptotically stable: by crossing the bifurcation value ($\mu = 0$), we keep unchanged the number of the equilibrium points, but their qualitative behavior changes. In Fig.4.4.3 the transcritical bifurcation diagram is reported: like in the saddle-node bifurcation, the continuous line refers to the asymptotically stable equilibrium point, while the dotted line refers to the unstable equilibrium point. Differently from the saddle-node bifurcation, the transcritical bifurcation is known as *safe* (or *soft*). Indeed, there is no dramatic change in the qualitative behavior when the bifurcation parameter crosses its bifurcation value: there are always two equilibrium points, the smaller unstable, the greater asymptotically stable (see the vector field diagram of Fig.4.4.4). Moreover the bifurcation occurs when the two points coincide.

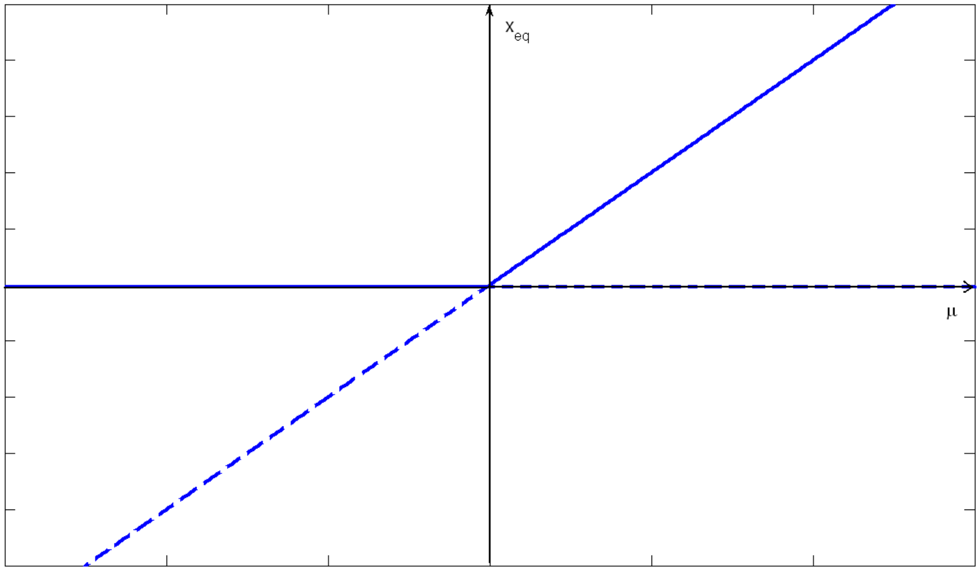

Fig. 4.4.3 - Transcritical bifurcation diagram.

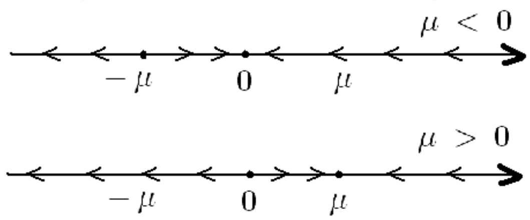

Fig. 4.4.4 - Vector field diagram for system (4.4.2).

c) **PITCHFORK BIFURCATION.** It occurs when, by crossing the bifurcation value, the only equilibrium point changes its qualitative behavior and a couple of two more equilibrium points (both with the same qualitative behavior) arise. For instance, consider the system:

$$\dot{x}(t) = \mu x(t) - x^3(t). \tag{4.4.3}$$

It admits only one equilibrium point (asymptotically stable) in $x_1 = 0$ when $\mu < 0$, and three equilibrium points in $x_1 = 0$ (unstable), $x_2 = \sqrt{\mu}$ (asymptotically stable) and $x_3 = -\sqrt{\mu}$ (asymptotically stable) when $\mu > 0$. In this case we deal with a *supercritical pitchfork bifurcation*, because the additional equilibrium points occur when the origin changes from stable to unstable. In Fig.4.4.5 the supercritical pitchfork bifurcation diagram is reported. A different pitchfork bifurcation occurs for:

$$\dot{x}(t) = \mu x(t) + x^3(t). \tag{4.4.4}$$

In this case we have the opposite qualitative behavior: there is only one equilibrium point (unstable) in $x_1 = 0$ when $\mu > 0$, and three equilibrium points in $x_1 = 0$ (asymptotically stable), $x_2 = \sqrt{-\mu}$ (unstable) and $x_3 = -\sqrt{-\mu}$ (unstable) when $\mu < 0$. The additional equilibrium points, im this case both unstable, occur when the origin changes from unstable to stable, and we call it a *subcritical pitchfork bifurcation*. In Fig.4.4.6 the subcritical pitchfork bifurcation diagram is reported. Note that the supercritical pitchfork bifurcation is *safe*, since a small change from $\mu < 0$ into $\mu > 0$ makes it so that the origin (which was asymptotically stable for $\mu < 0$) becomes unstable but bounded by a couple of asymptotically stable new equilibrium points, close to the origin for small positive values of μ. On the other hand, the subcritical pitchfork bifurcation is *dangerous*, since a small change from $\mu > 0$ into $\mu < 0$ makes it so that the origin (which was asymptotically stable for $\mu < 0$) becomes unstable and the evolution definitely diverges to infinity wherever is the initial state x_0.

d) **HOPF BIFURCATION.** All previous bifurcation are referred to the case of bifurcation values which make one eigenvalue of the Jacobian matrix (actually a scalar Jacobian in the aforementioned examples) cross the zero value. *Hopf bifurcations* refer to the crossing of the imaginary axis for a pair of eigenvalues, thus it is required at least a second order system: it consists in a bifurcation which changes a focus into a limit cycle. For instance, consider the second order system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big(\mu - x_1^2(t) - x_2^2(t)\big) - x_2(t),\\ \dot{x}_2(t) &= x_2(t)\big(\mu - x_1^2(t) - x_2^2(t)\big) + x_1(t).\end{aligned} \tag{4.4.5}$$

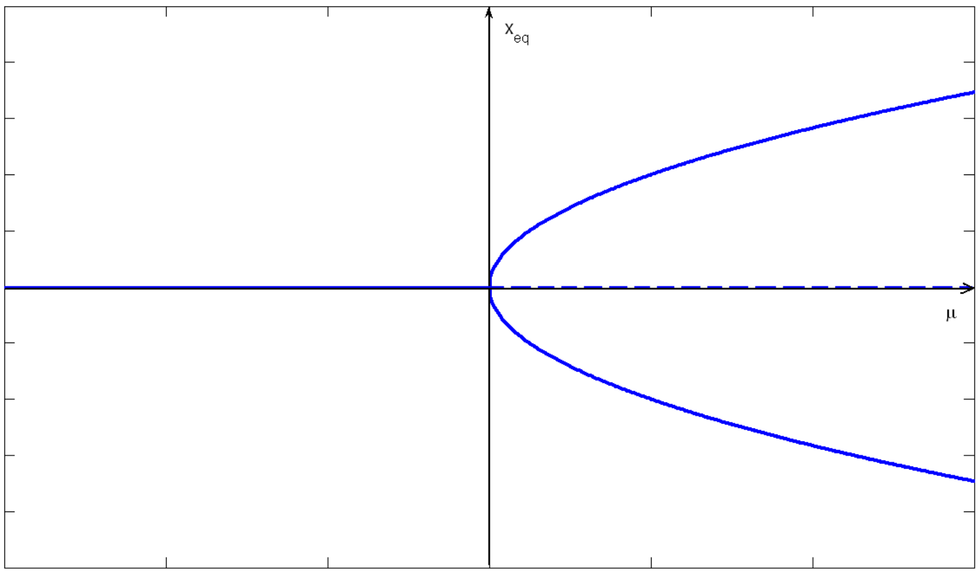

Fig. 4.4.5 - Supercritical pitchfork bifurcation diagram.

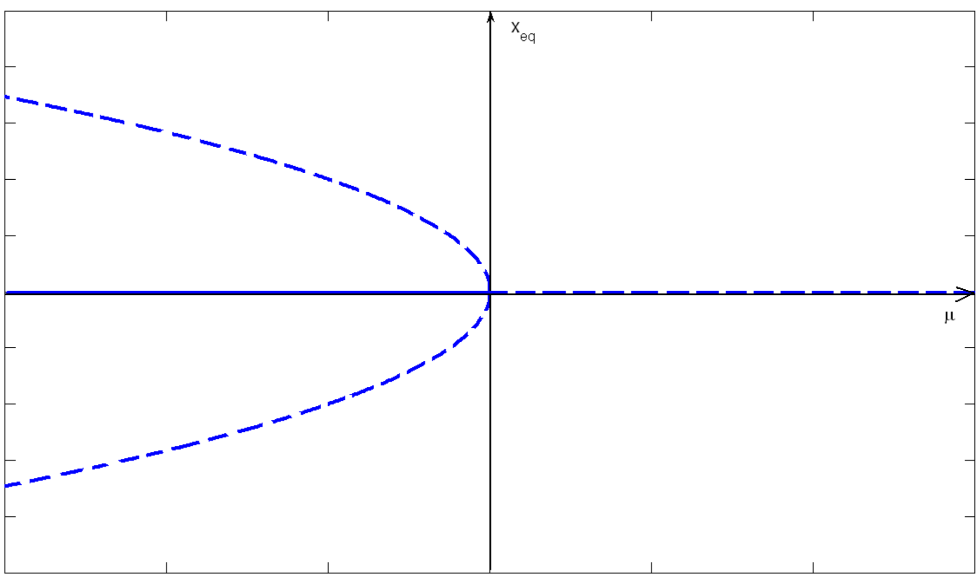

Fig. 4.4.6 - Subcritical pitchfork bifurcation diagram.

It can be shown that the origin is the only equilibrium point. For $\mu < 0$ we have a couple of complex eigenvalues in the negative real half-plane: an asymptotically stable focus. For $\mu > 0$ we have a couple of complex eigenvalue in the positive real half-plane: an unstable focus, which provides a stable limit cycle. This is the example of a *supercritical Hopf bifurcation*. In this case, the polar coordinates ($x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$) allow us to better understand the feature of the limit cycle. Indeed, in polar coordinates, system (4.4.5) becomes:

$$\begin{aligned}\dot{\rho}(t) &= \rho(\mu - \rho^2),\\ \dot{\theta}(t) &= 1,\end{aligned} \tag{4.4.6}$$

from which it follows that a nontrivial limit cycle occurs for $\mu > 0$, that it is a circumference with radius $\rho = \sqrt{\mu}$, and that it is a stable limit cycle, since the Jacobian $J(\rho) = \mu - 3\rho^2$ is negative when computed in $\rho = \sqrt{\mu}$. In Fig.4.4.7 the supercritical Hopf bifurcation diagram is reported: note that on the y-axis the radius of the limit cycle is reported, with filled circles as markers of the stable limit cycle.

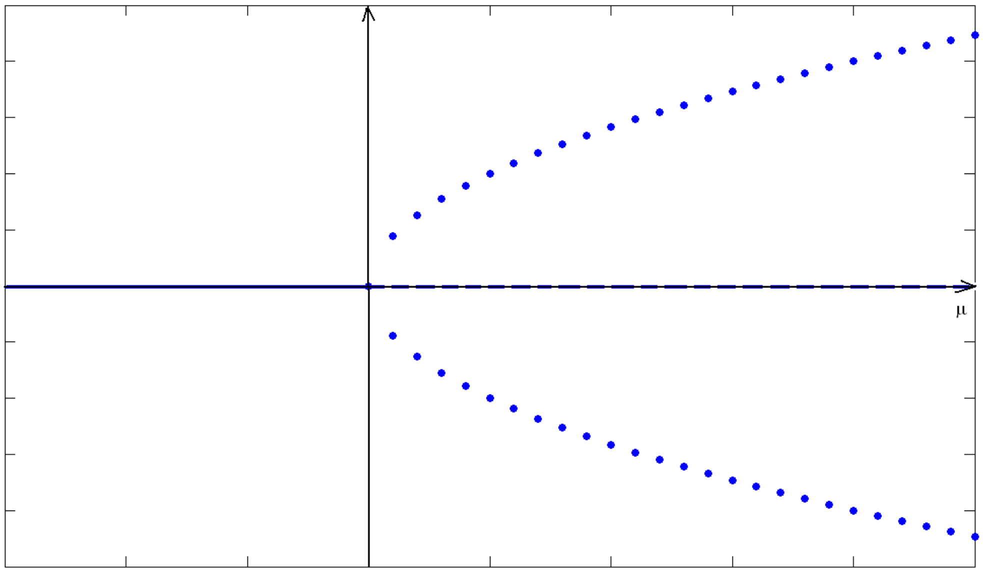

Fig. 4.4.7 - Supercritical Hopf bifurcation diagram.

It has to be stressed that the supercritical Hopf bifurcation is safe. Indeed, when approaching the bifurcation value from positive decreasing values of the bifurcation parameter μ, we have the occurrence of a stable limit cycle whose radius reduces uniformly with μ; thus, when μ becomes negative, providing an asymptotically stable focus, there is a very little change in the vector field diagram, as it can be appreciated from Fig.4.4.8, where decreasing values of μ are considered from the greater positive (Fig.4.4.8-A) to the smaller positive (Fig.4.4.8-C) and finally to the negative value (Fig.4.4.8-D).
A different Hopf bifurcation occurs for:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big(\mu + x_1^2(t) + x_2^2(t) - \big(x_1^2(t) + x_2^2(t)\big)^2\Big) - x_2(t), \\ \dot{x}_2(t) &= x_2(t)\Big(\mu + x_1^2(t) + x_2^2(t) - \big(x_1^2(t) + x_2^2(t)\big)^2\Big) + x_1(t). \end{aligned} \tag{4.4.7}$$

It can be shown that the origin is the only equilibrium point. Putting (4.4.7) into polar coordinates, it becomes:

$$\begin{aligned} \dot{\rho}(t) &= \rho(\mu + \rho^2 - \rho^4), \\ \dot{\theta}(t) &= 1, \end{aligned} \tag{4.4.8}$$

from which it follows that:

- for $\mu > 0$ the origin is unstable and there exist a stable limit cycle, which is a circumference with radius: $\rho_1 = \frac{1+\sqrt{1+4\mu}}{2}$;
- for $-\frac{1}{4} < \mu < 0$ the origin changes its qualitative behavior from unstable to stable and one more unstable limit cycle occurs, with radius $\rho_2 = \frac{1-\sqrt{1+4\mu}}{2}$ smaller that ρ_1; the limit cycle with radius ρ_1 is still stable;
- for $\mu < -\frac{1}{4}$ both limit cycle vanish: there is only asymptotically stable equilibrium point.

The bifurcation diagram is depicted in Fig.4.4.9, where the unstable limit cycle is represented with empty circles as markers. This is a *subcritical Hopf bifurcation* and its qualitative behavior is *dangerous*. For instance, consider an initial state which has a distance from the origin just a little bit greater that $\rho = -1/4$: for values of parameter μ immediately lower than $-1/4$ we have a stable focus, that means convergence to the origin (see Fig.4.4.10-A); for values of parameter μ immediately greater than $-1/4$

we have the occurrence of a stable limit cycle, with nontrivial radius, that means convergence to the limit cycle instead of convergence to the equilibrium point (see Fig.4.4.10-B).

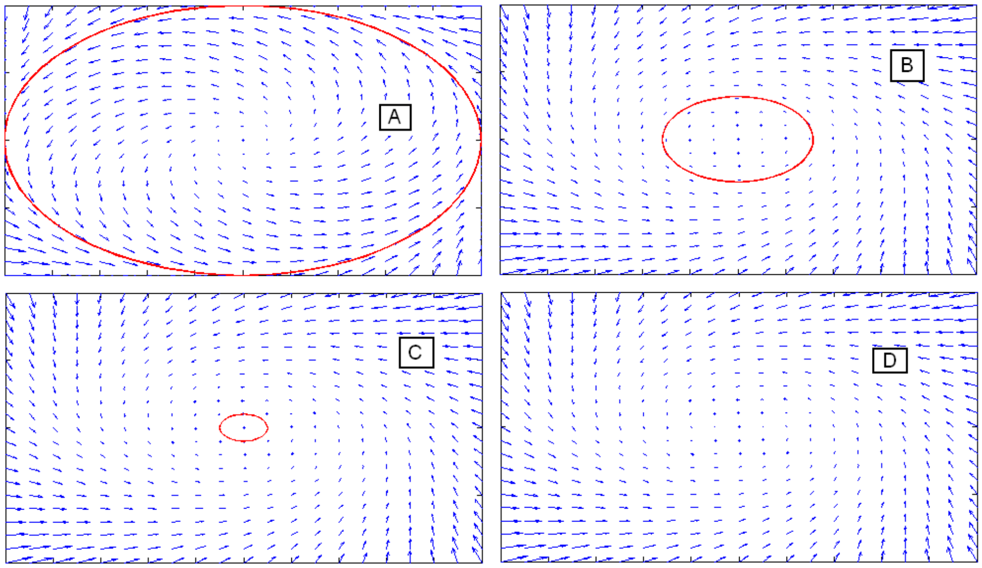

Fig. 4.4.8 - Supercritical Hopf bifurcation: vector field diagrams.

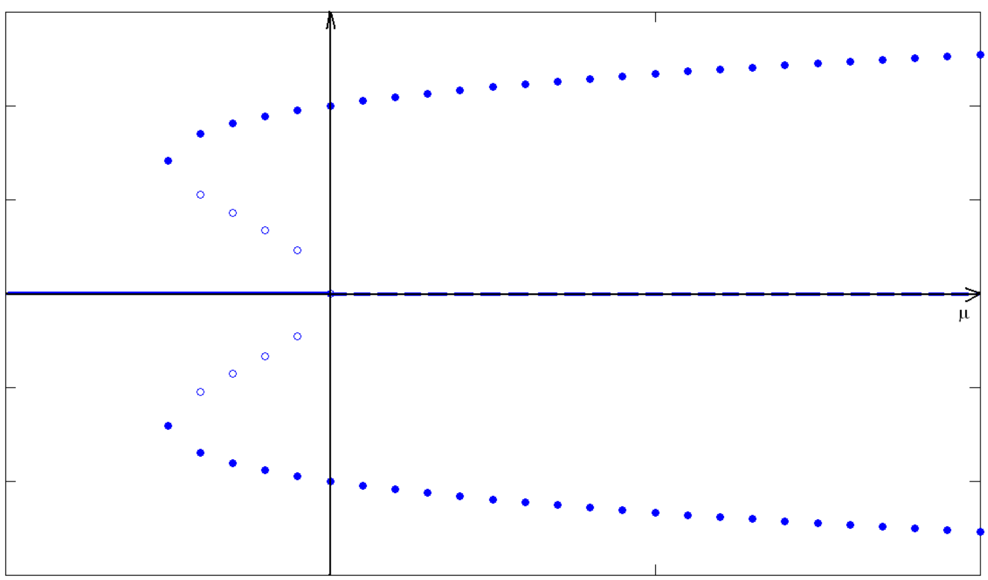

Fig. 4.4.9 - Subcritical Hopf bifurcation diagram.

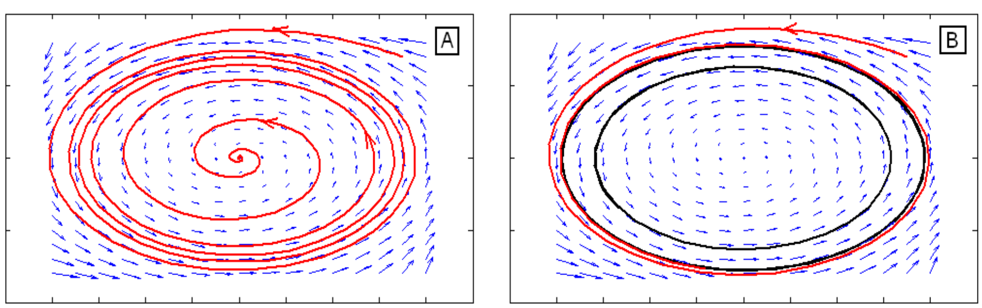

Fig. 4.4.10 - Subcritical Hopf bifurcation: vector field diagram.

5 - STABILITY CRITERIA

5.1 - LIAPUNOV CRITERION

This section is devoted to provide criteria to establish the stability of equilibrium points. Before stating the main criteria, the following definitions are required.

Definition 5.1.1. *A function $V : I\!R^n \mapsto I\!R$ is*
– *positive definite in a neighborhood I_ρ of a point $\bar{x}$ if:*

$$V(\bar{x}) = 0 \qquad \text{and} \qquad V(x) > 0 \qquad \forall x \in I_\rho(\bar{x}) \backslash \bar{x}; \tag{5.1.1}$$

– *positive semi-definite in a neighborhood I_ρ of a point $\bar{x}$ if:*

$$V(\bar{x}) = 0 \qquad \text{and} \qquad V(x) \geq 0 \qquad \forall x \in I_\rho(\bar{x}); \tag{5.1.2}$$

– *negative definite in a neighborhood I_ρ of a point $\bar{x}$ if $-V$ is positive definite;*
– *negative semi-definite in a neighborhood I_ρ of a point $\bar{x}$ if $-V$ is positive semi-definite.*

Theorem 5.1.2 (Liapunov Criterion). *Consider a time-invariant nonlinear system like in* (2.1.1)*, with x_e a given equilibrium point. Assume there exist a function $V : I\!R^n \mapsto I\!R$ and a neighborhood $I_\rho(x_e)$ of the equilibrium point such that:*
i) V admits a continuous derivative;
ii) V is positive definite;
iii-a) $\dot{V}$ is negative semi-definite.
Then x_e is stable. Moreover, if in $I_\rho(x_e)$:
iii-b) $\dot{V}$ is negative definite,
then x_e is locally asymptotically stable. Finally, if $I_\rho(x_e)$ can be extended to the whole state space $I\!R^n$ and V is radially unbounded, then x_e is globally asymptotically stable.

Proof. The proof is provided for second-order systems, that is $f : I\!R^2 \mapsto I\!R^2$. In this case, it is easy to verify that if V is continuous and positive definite in some $I_\rho(x_e)$, then there exists a neighborhood $I_{\tilde{\rho}}(x_e) \subseteq I_\rho(x_e)$ such that:
– any level surface $V_k = \{x \in I\!R^2 : \ V(x) = k\}$ is a closed orbit with x_e inside;
– if $k_1 < k_2$, then V_{k_1} is inside V_{k_2}.
– by decreasing $k \mapsto 0$, V_k collapses to the trivial closed orbit which consists of the equilibrium point: $V_0 = \{x_e\}$.

Thus, with no loss of generality, it will be assumed throughout the proof that previous three properties hold true for the neighborhood $I_\rho(x_e)$. Let $\varepsilon > 0$. Then, there exists a level surface V_k for some $k > 0$ such that V_k is inside $I_\varepsilon(x_e)$. Then there exists some $\delta > 0$ such that the neighborhood $I_\delta(x_e)$ is inside V_k. Now, let us choice x_0 inside $I_\delta(x_e)$. There will be a level surface associated to x_0 (named V_h) which will be necessarily inside V_k, since x_0 is inside V_k. According to item iii-a), by increasing the time t, the time derivative of V is non-positive, that means that V does not increase with time. Thus, the state evolution $x(t) = \varphi(t, x_0)$ associated to $V\big(x(t)\big)$ will be related to level surfaces necessarily inside the level surface V_h, which is inside V_k, which is inside $I_\varepsilon(x_e)$. Thus we have stability.

Moreover, if item iii-b) holds true, then, once the initial state x_0 is chosen in $I_\delta(x_0)$, by increasing the time t, the time derivative of V is strictly negative, that means that V monotonically decreases with time. Thus the state evolution $x(t) = \varphi(t, x_0)$ associated to $V\big(x(t)\big)$ will be related to level surfaces which collapse to the trivial closed orbit, consisting of the equilibrium point: $V_0 = \{x_e\}$. Thus we have asymptotic stability. We deal with local asymptotic stability, since the building of the Liapunov function $V(\cdot)$ is required in a neighborhood of the equilibrium point. In case of extending $I_\rho(x_e)$ to the whole state space, we still need a further property of radial unboundedness, that means:

$$\lim_{\|x\| \mapsto +\infty} |V(x)| = +\infty. \tag{5.1.3}$$

Otherwise, we could have not-closed level surfaces, with the result of loosing the attractivity property. $\diamondsuit$

Remark 5.1.3. As far as the computation of $\dot{V}(\cdot)$, note that:

$$\dot{V}\big(x(t)\big) = \frac{dV}{dx} \cdot \frac{dx}{dt} = \big(\nabla_x V\big) \cdot f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x), \tag{5.1.4}$$

that is the gradient of V along the trajectory of the evolution: $\dot{V}$ tells us of how $V(\cdot)$ changes by moving along the evolution of $x(t)$. •

Remark 5.1.4. Functions satisfying conditions i-ii) are usually called *candidate Liapunov functions*. It has to be stressed that the Liapunov Criterion provides just a sufficient condition. If we are keen enough to prove that a candidate Liapunov function satisfies also condition iii), then we have proved stability. But, in case we are not able to do it, or even if we prove that the candidate function is not a Liapunov function, then we are still wondering whether the equilibrium point is stable or not. Indeed, there is not a systematic method to design Liapunov function. •

Sometimes there are some natural candidates like energy functions in electrical or mechanical systems, but in any case it is always a matter of trial and error. A wide class of Liapunov candidates are *positive quadratic functions*. In general, a quadratic function can be written as:

$$V(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) = (x - \bar{x})^T Q (x - \bar{x}), \tag{5.1.5}$$

with Q a symmetric matrix. If Q were a non-symmetric matrix, then we could write:

$$(x-\bar{x})^T Q(x-\bar{x}) = (x-\bar{x})^T (Q_s + Q_a)(x-\bar{x}) = (x-\bar{x})^T Q_s (x-\bar{x}) + (x-\bar{x})^T Q_a (x-\bar{x}), \tag{5.1.6}$$

with

$$Q_s = \frac{Q + Q^T}{2}, \qquad Q_a = \frac{Q - Q^T}{2}. \tag{5.1.7}$$

Since Q_a is an anti-symmetric matrix, it is: $(x-\bar{x})^T Q_a (x-\bar{x}) = 0$, $\forall x, \bar{x}$; therefore any quadratic function $(x-\bar{x})^T Q(x-\bar{x})$ is equal to the quadratic function associated to the symmetric part Q_s of matrix Q. In the sequel, positive-definiteness of a quadratic function will be referred to positive-definiteness of the associated symmetric matrix. Thus the point is: when a symmetric matrix is positive-definite ? Having the possibility of computing the eigenvalues, a symmetric matrix is positive-definite/positive-semidefinite if, and only if, all its eigenvalues are strictly positive/non-negative. Otherwise, one can use the following criterion.

Theorem 5.1.5 (Sylvester Criterion). *A symmetric matrix;*

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}, \tag{5.1.8}$$

is positive-definite if, and only if, all the following determinants are positive:

$$M_k = \det \begin{bmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & \ddots & \vdots \\ q_{k1} & \cdots & q_{kk} \end{bmatrix} > 0, \qquad k = 1, \ldots, n. \tag{5.1.9}$$

♢

Once a symmetric positive matrix Q is chosen, if x_e is the equilibrium point to be investigated, the following quadratic function:

$$V(x) = (x - x_e)^T Q (x - x_e) \tag{5.1.10}$$

is a typical Liapunov candidate, being positive-definite in the whole state space $I\!R^n$ and also radially unbounded. Unfortunately, not always quadratic functions reveal to be Liapunov functions, as it happens for the following example.

Example 5.1.6 (The prey-predator model). Let us consider the following second order ODE:

$$\begin{cases} \dot{x}_1(t) = -ax_1(t) + bx_1(t)x_2(t), \\ \dot{x}_2(t) = cx_2(t) - dx_1(t)x_2(t), \end{cases} \qquad a, b, c, d > 0. \tag{5.1.11}$$

$x_1(t)$ and $x_2(t)$ denote predator and prey concentrations in a closed environment, where predators are convicted to extinction without preys (with an exponential decay rate equal to a), whilst preys would indefinitely increase without predators (with exponential rate c): assumption of infinite food for preys. It is the easiest version of the well known *Lotke-Volterra prey-predator model*. It readily appears that the origin is an equilibrium point which is unstable (it is a saddle node, actually). Indeed, this system has one more (more interesting!) equilibrium point, as it comes out by solving the algebraic system:

$$\begin{cases} -ax_1 + bx_1x_2 = 0, \\ cx_2 - dx_1x_2 = 0, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \bar{x}_1 = c/d, \\ \\ \bar{x}_2 = a/b. \end{cases} \tag{5.1.12}$$

After some computations, it comes out that a quadratic Liapunov candidate is not useful to investigate its stability; thus we will consider the following Liapunov candidate:

$$V(x) = \alpha\left(x_1 - \bar{x}_1 - \bar{x}_1 \ln\frac{x_1}{\bar{x}_1}\right) + \beta\left(x_2 - \bar{x}_2 - \bar{x}_2 \ln\frac{x_2}{\bar{x}_2}\right), \qquad \alpha, \beta > 0. \tag{5.1.13}$$

As a first step let us verify that in fact it is a Liapunov candidate: indeed, $V(\cdot)$ vanishes in $(\bar{x}_1, \bar{x}_2)$. Moreover the equilibrium point is a local minimum, since it also vanishes the gradient $\nabla_x V$:

$$\frac{\partial V}{\partial x_1} = \alpha\left(1 - \frac{\bar{x}_1}{x_1}\right), \qquad \frac{\partial V}{\partial x_2} = \beta\left(1 - \frac{\bar{x}_2}{x_2}\right), \tag{5.1.14}$$

and makes positive-definite the Hessian matrix:

$$\frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} \alpha\dfrac{\bar{x}_1}{x_1^2} & 0 \\ 0 & \beta\dfrac{\bar{x}_2}{x_2^2} \end{bmatrix} \qquad \Longrightarrow \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{(\bar{x}_1, \bar{x}_2)} = \begin{bmatrix} \dfrac{\alpha}{\bar{x}_1} & 0 \\ 0 & \dfrac{\beta}{\bar{x}_2} \end{bmatrix}. \tag{5.1.15}$$

Therefore, there exists a neighborhood $I_\rho(\bar{x})$ for which $V(x)$ is positive-definite: $V(x)$ is a Liapunov candidate. Let us compute $\dot{V}(x)$:

$$\begin{aligned} \dot{V}(x) &= \frac{\alpha(x_1 - \bar{x}_1)(-ax_1 + bx_1x_2)}{x_1} + \frac{\beta(x_2 - \bar{x}_2)(cx_2 - dx_1x_2)}{x_2} \\ &= \alpha(x_1 - \bar{x}_1)(-a + bx_2) + \beta(x_2 - \bar{x}_2)(c - dx_1) \\ &= \alpha b(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) - \beta d(x_2 - \bar{x}_2)(x_1 - \bar{x}_1). \end{aligned} \tag{5.1.16}$$

By setting $\alpha = \beta d/b$ we have $\dot{V}(x) \equiv 0$, which is clearly a negative semi-definite function, which guarantees stability. More in details, the proposed Liapunov function is a *first integral of motion*, that means whatever is the initial state x_0, we have $V\big(x(t)\big) \equiv V(x_0)$ for any $t \geq 0$. •

The Liapunov criterion has the following characterization for time-invariant linear systems.

Theorem 5.1.7. *Consider a linear system as the one defined in* (3.1.1). *The system is asymptotically stable if, and only if, for any symmetric, positive-definite matrix $P \in I\!R^{n\times n}$, the matrix equation (Liapunov equation):*

$$A^T Q + QA = -P \tag{5.1.17}$$

in the unknown $Q \in I\!R^{n\times n}$ admits a unique, symmetric positive-definite solution.

Proof. ($\Longrightarrow$) Let us assume the asymptotic stability. Then, all the eigenvalues of matrix A have strictly negative real part. Define matrix Q has:

$$Q = \int_0^{+\infty} e^{A^T t} P e^{At} dt. \tag{5.1.18}$$

Such an integral is properly defined. Indeed, due to the stability property, according to the spectral decomposition, matrix A can be decomposed in the finite sum of strictly decreasing exponentials, possibly multiplied per polynomials, thus allowing a finite sum for the integral. Matrix Q is clearly symmetric, since matrix P is:

$$Q^T = \int_0^{+\infty} \left(e^{At}\right)^T P^T \left(e^{A^T t}\right)^T dt = Q, \tag{5.1.19}$$

and, moreover, it is positive definite, since matrix P is:

$$x^T Q x = \int_0^{+\infty} x^T e^{A^T t} P e^{At} x dt = \int_0^{+\infty} \left(e^{At} x\right)^T P \left(e^{At} x\right) dt. \tag{5.1.20}$$

Finally, matrix Q is the unique solution of the Liapunov equation. Indeed the Liapunov equation admits a unique solution. To show such a preliminary item, we will use a Lemma concerning the more generic matrix equation in the unknown Q:

$$RQ - QS = T, \qquad R, S, T \in I\!R^{n \times n}. \tag{5.1.21}$$

The Lemma states that if R and S have no common eigenvalues, there exists a unique solution. The Liapunov equation belongs to the above mentioned class of equations, with $R = A^T$ and $S = -A$; thus the eigenvalues of R belong to the strictly negative real half-plane, being the same of the asymptotically stable matrix A, and the eigenvalues of S belong to the strictly positive real half-plane, being the opposite of matrix A. Thus: no common eigenvalues implies a unique solution. To show matrix Q is in fact the solution of the Liapunov equation, we finally need to substitute it in (5.1.17):

$$\begin{aligned} &A^T \left(\int_0^{+\infty} e^{A^T t} P e^{At} dt \right) + \left(\int_0^{+\infty} e^{A^T t} P e^{At} dt \right) A = \int_0^{+\infty} A^T e^{A^T t} P e^{At} dt + \int_0^{+\infty} e^{A^T t} P e^{At} A dt \\ &= \int_0^{+\infty} \left(A^T e^{A^T t} P e^{At} + e^{A^T t} P e^{At} A \right) dt = \int_0^{+\infty} \frac{d}{dt} \left(e^{A^T t} P e^{At} \right) dt \\ &= \lim_{t \mapsto +\infty} e^{A^T t} P e^{At} - \lim_{t \mapsto 0^+} e^{A^T t} P e^{At} = -P. \end{aligned} \tag{5.1.22}$$

($\Longleftarrow$) Let us assume the Liapunov equation admits a symmetric, positive-definite solution. Then, the following Liapunov candidate is given to evaluate the asymptotic stability of the origin:

$$V(x) = x^T Q x. \tag{5.1.23}$$

By computing the time derivative:

$$\dot{V}(x) = \dot{x}^T Q x + x^T Q \dot{x} = x^T A^T Q x + x^T Q A x = x^T (A^T Q + QA) x = -x^T P x \tag{5.1.24}$$

which is a quadratic negative-definite function because P is a symmetric positive-definite matrix. $\Diamond$

5.2 - LINEARIZATION CRITERION

We have already dealt with linearization in Section 4.3. In this Section we will show how it does reveal to be powerful in investigating stability.

Theorem 5.2.1 (Linearization Criterion). *Consider a time-invariant nonlinear system like the one in* (2.1.1) *with* x_e *a given equilibrium point. Consider the Jacobian matrix of function* $f(\cdot)$ *as defined in* (4.3.2). *Then:*

i) if all the eigenvalues of the Jacobian matrix have strictly negative real part, the equilibrium point is locally asymptotically stable;

ii) if there is even just one eigenvalue with strictly positive real part then the equilibrium point is unstable, regardless to the sign of all the other eigenvalues

iii) if all the eigenvalues have strictly negative real part, except for one (or more) with null real part, then nothing can be said according to this criterion.

Proof of Item i). Consider the displacement $z(t) = x(t) - x_e$ and its dynamic equation obtained in (4.3.4) and below reported:

$$\dot{z}(t) = J(x_e)z(t) + h\big(z(t)\big), \qquad \text{with} \qquad \lim_{\|z\|\mapsto 0} \frac{\|h\big(z(t)\big)\|}{\|z\|} = 0. \tag{5.2.1}$$

If matrix $J(x_e)$ is asymptotically stable then, according to Theorem 5.1.7, there exists a unique, symmetric, positive-definite solution to the Liapunov equation:

$$J^T(x_e)Q + QJ(x_e) = -P \tag{5.2.2}$$

for any given symmetric, positive-definite matrix P. Let us use Q to consider the following Liapunov candidate function:

$$V(x) = (x - x_e)^T Q(x - x_e) = z^T Qz, \tag{5.2.3}$$

from which:

$$\begin{aligned} \dot{V}(x) &= \dot{z}^T Qz + z^T Q\dot{z} = z^T J^T(x_e)Qz + h^T(z)Qz + z^T QJ(x_e)z + z^T Qh(z) \\ &= z^T \big(J^T(x_e)Q + QJ(x_e)\big)z + 2h^T(z)Qz = -z^T Pz + 2h^T(z)Qz. \end{aligned} \tag{5.2.4}$$

Now:

$$\lim_{\|z\|\mapsto 0} \frac{|h(z)Qz|}{\|z\|^2} = 0 \qquad \text{but} \qquad \lim_{\|z\|\mapsto 0} \frac{|z^T Pz|}{\|z\|^2} > 0. \tag{5.2.5}$$

It means there exists a positive radius ρ such that for $x \in I_\rho(x_e)$, it is $\dot{V}(x) < 0$, which implies local asymptotic stability. $\diamondsuit$