QUALITATIVE BEHAVIOR OF SOLUTIONS:
Equilibrium Points and Stability
Linear Systems

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Time-invariant, nonlinear systems (ODE models):

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0, \qquad x(t) \in \mathbb{R}^n, \qquad f : \mathbb{R}^n \mapsto \mathbb{R}^n.$$

- The unique solution to the Chauchy problem will be denoted by:

$$x(t) = \varphi(t, x_0)$$

  where $\varphi : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is the **state-transition map**.

- x_0 is the initial state: it univocally determines the evolution of the state variables x

- **Equilibrium points** are such that no motion occurs if they are chosen as initial state:

$$\varphi(t, x_e) = x_e, \qquad \forall t \geq 0$$

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- By definition, the equilibrium points vanish the time-derivative:

$$f(x_e) = 0$$

- In case of a generic **nonlinear** system we can have:
  - no equilibrium points at all: $f(x) = 0$ has no solutions
  - a unique equilibrium point: $f(x) = 0$ admits a unique solution
  - isolated equilibrium points: $f(x) = 0$ admits a discrete number of solutions
  - infinite equilibrium points: $f(x) = 0$ admits infinite solutions
- In case of **linear** systems, $f(x) = Ax$, we can have:
  - the origin is always an equilibrium point
  - if $\mathrm{rank}(A) = n$, the origin is the unique equilibrium point
  - if $\mathrm{rank}(A) = r < n$, there exist ∞^{n-r} (uncountable) equilibrium points
  - there can never be isolated equilibrium points, unless the origin

Equilibrium points

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- **Stability.** The equilibrium point x_e is *stable* if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|x_0 - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \varepsilon, \quad \forall t \geq 0$$

In case of **linear** systems, $f(x) = Ax$, the stability of a given equilibrium point implies and is implied by the stability of the origin

- **Attractivity.** The equilibrium point x_e is:
  - *locally attractive* if: $\exists \eta > 0 : \quad \|x_0 - x_e\| < \eta \quad \Longrightarrow \quad \|x(t) - x_e\| \mapsto 0$
  - *globally attractive* if: $\forall x_0 \in \mathbb{R}^n, \quad \textit{it is} : \quad \|x(t) - x_e\| \mapsto 0$

Attractivity can occur only if the equilibrium point is **isolated**

In case of **linear** systems, $f(x) = Ax$, only the origin can be attractive

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- **Asymptotic Stability.** The equilibrium point x_e is *locally/globally asymptotically stable* if it is stable and locally/globally attractive

Global asymptotic stability can occur only when the equilibrium point is unique

- **Exponential Stability.** The equilibrium point x_e is *exponentially stable* if:

$$\exists \alpha > 0 : \quad \forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|x_0 - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \varepsilon \cdot e^{-\alpha t}$$

In case of **linear** systems, $f(x) = Ax$, asymptotical stability is always **global** and **exponential**

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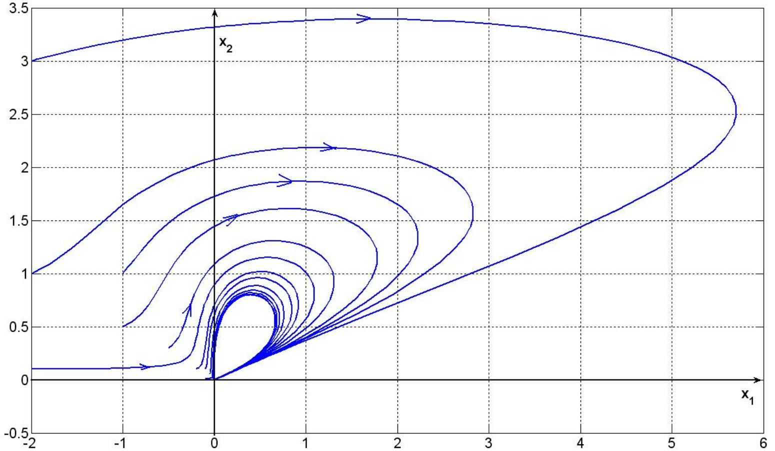
Time-invariant, nonlinear systems (ODE models):

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2))} \qquad \dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2))}$$



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$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n\times n}$$

- The solution is a linear transformation of the initial state:

$$x(t) = \Phi(t)x_0 \qquad \Phi(t) \quad \textbf{is the state transition matrix}$$

- $\Phi(t)$ obeys the following matricial Chauchy problem

$$\dot{\Phi}(t) = A\Phi(t), \qquad \Phi(0) = I_n$$

The solution is the **exponential matrix**:

$$\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

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- $e^{At}\Big|_{t=0} = \left[I_n + At + \frac{A^2t^2}{2} + \cdots\right]_{t=0} = I_n$
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- Further property (**semigroup**):

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Linear systems evolution

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- Spectrum of matrix A: $\{\lambda_1, \ldots, \lambda_n\}$ $\quad$ n distinct eigenvalues
- The associated eigenvectors $\{u_1, \ldots, u_n\}$ are a basis for the state space:

$$U = \left[u_1 \cdots u_n\right] \qquad \Longrightarrow \qquad A = U\Lambda U^{-1} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix}$$

- **Spectral decomposition** of matrix A:

$$A = U\Lambda V = \sum_{i=1}^{n} \lambda_i u_i v_i^T, \qquad v_i^T \quad \text{are the rows of matrix } U^{-1}$$

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