QUALITATIVE BEHAVIOR OF SOLUTIONS: Equilibrium Points and Stability Linear Systems

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Equilibrium points

Time-invariant, nonlinear systems (ODE models):

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad x(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \mapsto \mathbb{R}^n. \]

- The unique solution to the Cauchy problem will be denoted by:
  \[ x(t) = \varphi(t, x_0) \]
  where \( \varphi : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n \) is the state-transition map.

- \( x_0 \) is the initial state: it univocally determines the evolution of the state variables \( x \)

- Equilibrium points are such that no motion occurs if they are chosen as initial state:
  \[ \varphi(t, x_e) = x_e, \quad \forall t \geq 0 \]
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- The unique solution to the Chauchy problem will be denoted by:

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- By definition, the equilibrium points vanish the time-derivative:
  \[ f(x_e) = 0 \]

- In case of a generic **nonlinear** system we can have:
  - no equilibrium points at all: \( f(x) = 0 \) has no solutions
  - a unique equilibrium point: \( f(x) = 0 \) admits a unique solution
  - isolated equilibrium points: \( f(x) = 0 \) admits a discrete number of solutions
  - infinite equilibrium points: \( f(x) = 0 \) admits infinite solutions

- In case of **linear** systems, \( f(x) = Ax \), we can have:
  - the origin is always an equilibrium point
  - if \( \text{rank}(A) = n \), the origin is the unique equilibrium point
  - if \( \text{rank}(A) = r < n \), there exist \( \infty^{n-r} \) (uncountable) equilibrium points
  - there can never be isolated equilibrium points, unless the origin

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### Time-invariant, nonlinear systems (ODE models):

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- **Stability.** The equilibrium point \( x_e \) is *stable* if:
  \[ \forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \implies \|x(t) - x_e\| < \varepsilon, \quad \forall t \geq 0 \]

In case of **linear** systems, \( f(x) = Ax \), the stability of a given equilibrium point implies and is implied by the stability of the origin.

- **Attractivity.** The equilibrium point \( x_e \) is:
  - *locally attractive* if: \( \exists \eta > 0 : \|x_0 - x_e\| < \eta \implies \|x(t) - x_e\| \rightarrow 0 \)
  - *globally attractive* if: \( \forall x_0 \in \mathbb{R}^n, \text{ it is: } \|x(t) - x_e\| \rightarrow 0 \)

Attractivity can occur only if the equilibrium point is *isolated*.

In case of **linear** systems, \( f(x) = Ax \), only the origin can be attractive.
Stability

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- **Asymptotic Stability.** The equilibrium point \( x_e \) is *locally/globally asymptotically stable* if it is stable and locally/globally attractive.

Global asymptotic stability can occur only when the equilibrium point is unique.

- **Exponential Stability.** The equilibrium point \( x_e \) is *exponentially stable* if:

  \[ \exists \alpha > 0 : \quad \forall \varepsilon > 0, \ \exists \delta > 0 : \quad \|x_0 - x_e\| < \delta \implies \|x(t) - x_e\| < \varepsilon \cdot e^{-\alpha t} \]

In case of linear systems, \( f(x) = Ax \), asymptotical stability is always *global* and *exponential*.
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Stability: example of attractivity without stability

Time-invariant, nonlinear systems (ODE models):

\[
\dot{x}_1 = \frac{x_1^2 (x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2))} \\
\dot{x}_2 = \frac{x_2^2 (x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2))}
\]
**Time-invariant, linear systems (ODE models):**

\[
\dot{x}(t) = Ax(t), \quad x(0) = x_0 \quad x(t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}
\]

- The solution is a linear transformation of the initial state:
  \[
x(t) = \Phi(t)x_0 \quad \Phi(t) \text{ is the state transition matrix}
\]

- \(\Phi(t)\) obeys the following matricial Cauchy problem
  \[
  \dot{\Phi}(t) = A\Phi(t), \quad \Phi(0) = I_n
  \]

The solution is the **exponential matrix**:

\[
\Phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}
\]
Linear systems evolution

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**Explicit solution to:**

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\[ \left. e^{At} \right|_{t=0} = \left[ I + At + \frac{A^2 t^2}{2} + \cdots \right]_{t=0} = I_n \]

\[ \frac{d}{dt} \left[ e^{At} \right] = \frac{d}{dt} \left[ \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right] = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A \sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = Ae^{At} \]

**Further property (semigroup):**

\[ \Phi(t_1 + t_2) = \Phi(t_1) \cdot \Phi(t_2) \]

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**Linear systems evolution**

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Linear systems evolution: **spectral decomposition**

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- Spectrum of matrix \( A \): \( \{\lambda_1, \ldots, \lambda_n\} \) \( n \) distinct eigenvalues
- The associated eigenvectors \( \{u_1, \ldots, u_n\} \) are a basis for the state space:

\[
U = [u_1 \cdots u_n] \quad \implies \quad A = U\Lambda U^{-1} \quad \Lambda = \begin{bmatrix}
\lambda_1 & & \\
& \ddots & \\
& & \lambda_n
\end{bmatrix}
\]

- **Spectral decomposition** of matrix \( A \):

\[
A = U\Lambda V = \sum_{i=1}^{n} \lambda_i u_i v_i^T, \quad v_i^T \text{ are the rows of matrix } U^{-1}
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### Linear systems evolution: **spectral decomposition**

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