QUALITATIVE BEHAVIOR OF SOLUTIONS:
Natural Modes
Qualitative Behavior of Linear Systems

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Linear systems evolution: **exponential matrix**

Time-invariant, linear systems (ODE models):

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n\times n}$$

Solution: the state-transition matrix is the exponential of matrix A

$$x(t) = e^{At}x_0 \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{t!}$$

How to compute the exponential matrix: Spectral Decomposition:

$$A = \sum_{i=1}^{n} \lambda_i u_i v_i^T$$

- $A^k = U\Lambda^k U^{-1}$

$$A^{k+1} = A \cdot U\Lambda^k U^{-1} = U\Lambda U^{-1} \cdot U\Lambda^k U^{-1} = U\Lambda \cdot \Lambda^k U^{-1} = U\Lambda^{k+1} U^{-1}$$

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- By computation:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{U\Lambda^k U^{-1} t^k}{k!} = U\left(\sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}\right)U^{-1} = Ue^{\Lambda t}U^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} \qquad \Longrightarrow \qquad e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T$$

Linear systems evolution: **natural modes**

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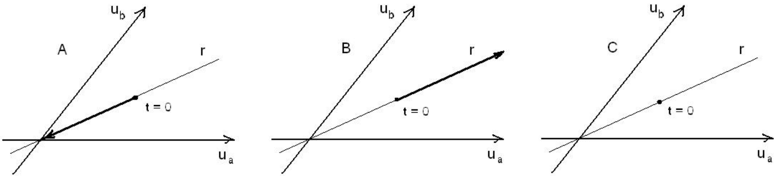
- Solution: $x(t) = Ue^{\Lambda t}U^{-1}x_0 = \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T x_0$
- Decomposition of the initial state x_0:

$$x_0 = c_1 u_1 + \cdots + c_n u_n = U\alpha, \qquad \alpha = [c_1 \ \cdots \ c_n]^T$$

- **Natural modes** of the system:

$$x(t) = Ue^{\Lambda t}U^{-1}U\alpha = Ue^{\Lambda t}\alpha = \sum_{i=1}^{n} e^{\lambda_i t} c_i u_i$$

- ***Aperiodic* natural modes**: $\lambda \in \mathbb{R}$: $\quad e^{\lambda t} cu$



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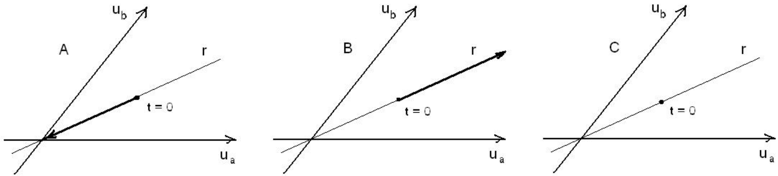

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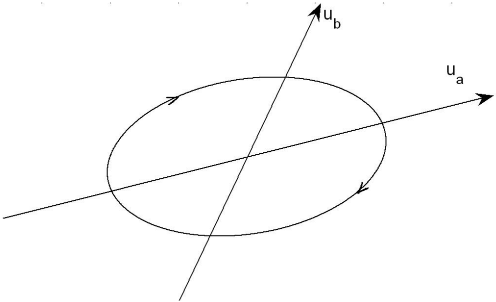
- ***Periodic* natural modes**: $\lambda_{1/2} = \pm j\omega, \quad c_{1/2} = \rho e^{\pm j\theta}, \quad u_{1/2} = u_a \pm j u_b$

$$e^{j\omega t} c u + e^{-j\omega t} c^\star u^\star = 2\rho\Big(\cos(\omega t + \theta) u_a - \sin(\omega t + \theta) u_b\Big)$$



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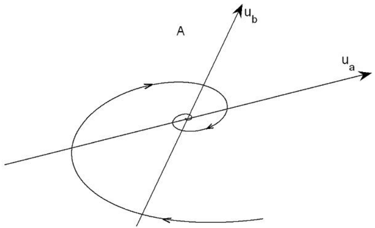
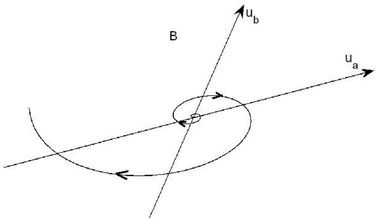
- ***Pseudo-periodic*** **natural modes**:

$$\lambda_{1/2} = \alpha \pm j\omega \qquad c_{1/2} = \rho e^{\pm j\theta} \qquad u_{1/2} = u_a \pm j u_b$$

$$e^{\alpha + j\omega t} c u + e^{\alpha - j\omega t} c^\star u^\star = 2\rho e^{\alpha t}\Big(\cos(\omega t + \theta) u_a - \sin(\omega t + \theta) u_b\Big)$$





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Explicit solution:

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} c_i u_i$$

- A linear system is **stable** if and only if all its eigenvalues have non-positive real part
- A linear system is **asymptotically stable** if and only if all its eigenvalues have (strictly) negative real part (**Hurwitz matrix**)
- A linear system is **unstable** if just one eigenvalue has positive real part
- Asymptotic stability for linear system is always **global** and **exponential**

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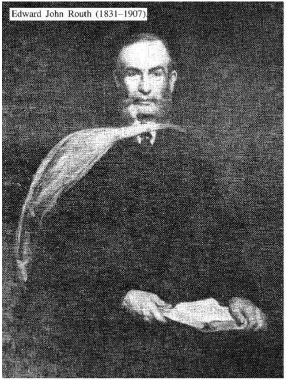
Stability for linear systems is all about the real part of the eigenvalues:

Routh-Hurwitz Criterion



Routh-Hurwitz Criterion

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$$

$$\begin{array}{c|ccccc}
n & \alpha_n & \alpha_{n-2} & \alpha_{n-4} & \alpha_{n-6} & \cdots \\
n-1 & \alpha_{n-1} & \alpha_{n-3} & \alpha_{n-5} & \alpha_{n-7} & \cdots \\
n-2 & a^2_{n-2} & a^2_{n-4} & a^2_{n-6} & \cdots & \\
n-3 & a^3_{n-3} & a^3_{n-5} & a^3_{n-7} & \cdots & \\
\vdots & \vdots & \vdots & & & \\
2 & a^{n-2}_2 & a^{n-2}_0 & & & \\
1 & a^{n-1}_1 & & & & \\
0 & a^n_0 & & & &
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$n-2$	a_{n-2}^2	a_{n-4}^2	a_{n-6}^2	$\cdots$	
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$\vdots$	$\vdots$	$\vdots$			
2	a_2^{n-2}	a_0^{n-2}			
1	a_1^{n-1}				
0	a_0^n				

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Routh-Hurwitz Criterion

$$p(\lambda) = \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 2$$

$$\begin{array}{c|ccc} 5 & 1 & 1 & 1 \\ 4 & 1 & 1 & 2 \\ 3 & 0 & -1 & \end{array}$$

$$\begin{array}{c|ccc} 5 & 1 & 1 & 1 \\ 4 & 1 & 1 & 2 \\ 3 & \varepsilon & -1 & \\ 2 & \frac{1}{\varepsilon} & 2 & \\ 1 & -1 & & \\ 0 & 2 & & \end{array}$$

$$a_2^3 = \frac{1+\varepsilon}{\varepsilon} = \frac{1}{\varepsilon}, \qquad a_1^4 = -1 - 2\varepsilon^2 = -1.$$

2 variations $\Longrightarrow$ 2 roots with positive real part

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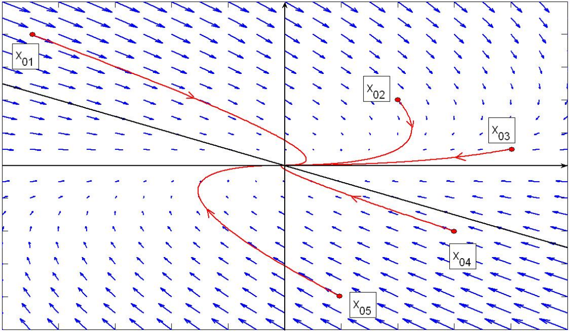
Time-invariant, linear systems (ODE models):

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in \mathbb{R}^2, \qquad A \in \mathbb{R}^{2\times 2}$$

Distinct real, negative eigenvalues: the origin is a **STABLE NODE**

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \qquad \lambda_1 < \lambda_2 < 0$$



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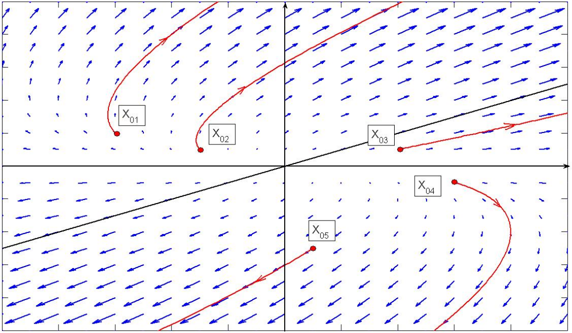
Distinct real, positive eigenvalues: the origin is a **UNSTABLE NODE**

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \qquad \lambda_1 > \lambda_2 > 0$$



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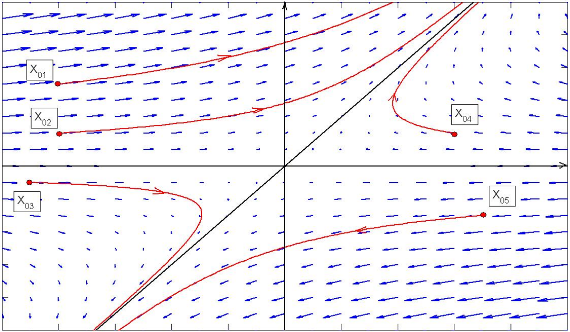
Distinct real, positive and negative eigenvalues: a **SADDLE NODE**

$$x(t) = e^{\lambda_1 t} c_1 u_1 + e^{\lambda_2 t} c_2 u_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \qquad \lambda_1 < 0 < \lambda_2$$



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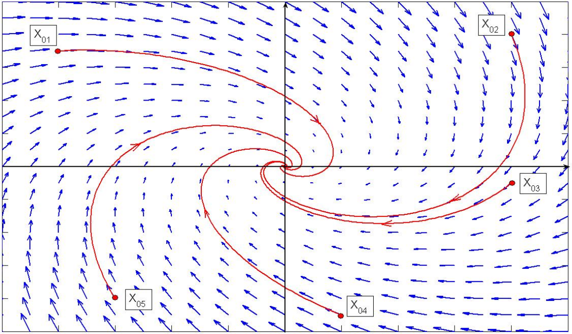
Complex eigenvalues, with negative real part: a **STABLE FOCUS**

$$x(t) = 2\rho e^{\alpha t}\Big(\cos(\beta t + \theta)u_\alpha - \sin(\beta t + \theta)u_\beta\Big), \qquad \alpha < 0$$



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Complex eigenvalues, with positive real part: an **UNSTABLE FOCUS**

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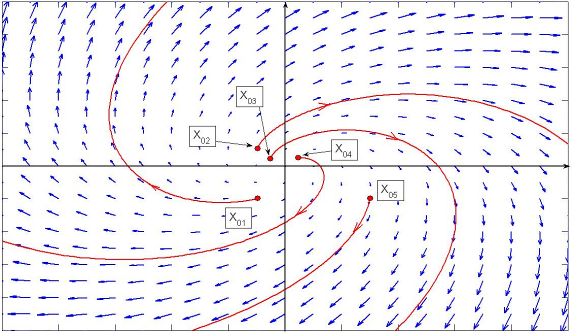

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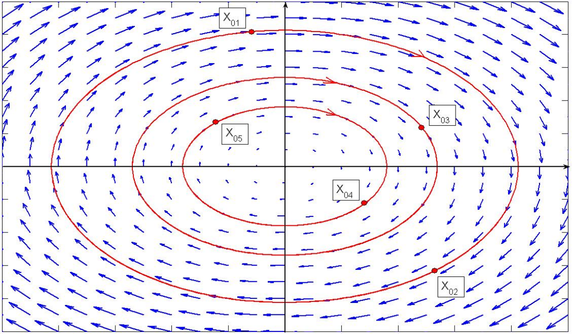
Complex eigenvalues, with null real part: a **CENTER**

$$x(t) = 2\rho\Big(\cos(\beta t + \theta)u_\alpha - \sin(\beta t + \theta)u_\beta\Big)$$



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Distinct real eigenvalues: one null, the other negative

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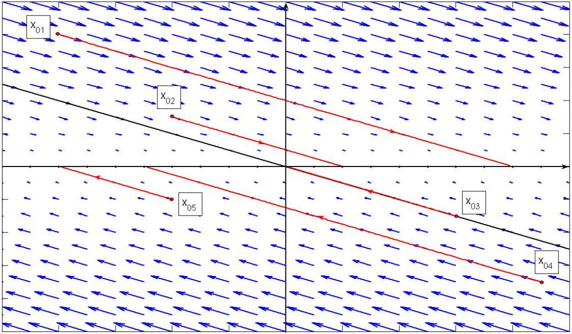

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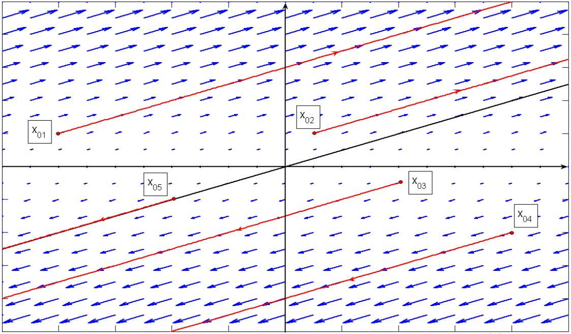
Distinct real eigenvalues: one null, the other positive

$$x(t) = c_1 u_1 + e^{\lambda t} c_2 u_2 \qquad \lambda > 0$$



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$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad x(t) \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n \times n}$$

- All distinct, *real*, *negative* eigenvalues:
  - the origin is a **STABLE NODE**
- All distinct, *real*, *positive* eigenvalues:
  - the origin is an **UNSTABLE NODE**
- All distinct, *real*, eigenvalues, *some positive*, *some negative*:
  - the origin is a **SADDLE NODE**
- All distinct eigenvalues, *some complex*, all with *negative* real part:
  - the origin is a **STABLE FOCUS**
- All distinct eigenvalues, *some complex*, all with *positive* real part:
  - the origin is an **UNSTABLE FOCUS**

Time-invariant, nonlinear systems (ODE models):

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0 \qquad x(t) \in \mathbb{R}^n, \qquad f : \mathbb{R}^n \mapsto \mathbb{R}^n$$

Taylor series expansion

$$\dot{x}(t) = f(x_e) + J(x_e)\big(x(t) - x_e\big) + h\big(x(t) - x_e\big)$$

$$J(x) = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \lim_{\|z\| \mapsto 0} \frac{\big\|h(\|z\|)\big\|}{\|z\|} = 0$$

- Displacement $z(t) = x(t) - x_e$ dynamics:

$$\dot{z}(t) = J(x_e)z(t) + h\big(z(t)\big)$$

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